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Antimagic Labeling of Some Path and Cycle Related Graphs

S. K. Vaidya [1] **and *N. B. Vyas*** [2]

[1]Department of Mathematics, Saurashtra University,
Rajkot – 360005, Gujarat, India. E-mail: samirkvaidya@yahoo.co.in

[2]Department of Mathematics, Atmiya Institute of Technology and Science,
Rajkot – 360005, Gujarat, India. E-mail: niravbvyas@gmail.com

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Abstract. An edge labeling of a graph is a bijection from $E(G)$ to the set $\{1,2,\dots, |E(G)|\}$. If for any two distinct vertices u and v, the sum of labels on the edges incident to u is different from the sum of labels on the edges incident to v then an edge labeling is called antimagic labeling. We investigate antimagic labeling for some path and cycle related graphs.

Keywords: Antimagic labeling, Antimagic graph, Middle graph, Total graph.

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

We begin with a finite, connected and undirected graph $G=(V(G),E(G))$ without loops and multiple edges. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the number of vertices and number of edges respectively. For any graph theoretic notation and terminology we rely upon Balakrishnan and Ranganathan [1].

A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a *vertex labeling* (an *edge labeling*).

According to Beineke and Hegde[2] labeling of discrete structure is a frontier between graph theory and theory of numbers. For an extensive survey of graph labeling as well as bibliographic references we refer to Gallian[3].

The concept of magic labeling was introduced during 1963 by Sedlacek[5]. A graph is said to be *magic* if it has a real-valued edge labeling such that;

(i) distinct edges have distinct non-negative labels;

(ii) the sum of the labels of the edges incident to a particular vertex is same for all the vertices.

An *antimagic labeling* of a graph G is a bijection from $E(G)$ to the set $\{1, 2, \dots, |E(G)|\}$ such that for any two distinct vertices u and v, the sum of the labels on edges incident to u is different from the sum of the labels on edges incident to v.

Hartsfield and Ringel[4] have introduced the concept of an antimagic graph in 1990. They proved that paths P_n $(n \leq 3)$, cycles, wheels, and complete graphs K_n $(n \leq 3)$ admit antimagic labeling. They have also conjectured that;

(i) all trees except K_2 are antimagic.

(ii) all connected graphs except K_2 are antimagic.

These two conjectures are still not settled.

The graphs obtained by switching of vertex in path P_n, cycle C_n, wheel W_n, helm H_n and fan f_n are proved to be antimagic by Vaidya and Vyas[6].

The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it. The *total graph* $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G. For a graph G the *splitting graph* $S'(G)$ is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v)=N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

2. Main Results

Theorem 2.1. Middle graph of path P_n is antimagic.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices and $e_1, e_2, \dots, e_n$ be the edges of path P_n and $G = M(P_n)$ be the middle graph of path P_n. According to the definition of middle graph $V(M(P_n)) = V(P_n) \cup E(P_n)$ and $E(M(P_n)) = \{v_ie_i\,;\ 1 \leq i \leq n-1,\ v_ie_{i-1}\,;\ 2 \leq i \leq n,\ e_ie_{i+1}\,;\ 1 \leq i \leq n-2\}$. Here $|V(G)| = 2n-1$ and $|E(G)| = 3n-4$. To define $f : E(G) \rightarrow \{1, 2, \dots, 3n-4\}$, we consider following two cases.

Case 1: $n = 3, 5$ and $n \equiv 0 \ (mod\ 2)$

For $1 \leq i \leq n-1$:

$f(v_ie_i) = 2i - 1$; $\quad f(e_iv_{i+1}) = 2i$;

For $1 \leq i \leq n-2$:

$f(e_ie_{i+1}) = 2(n-1) + i$;

Case 2: $n \equiv 1 \ (mod\ 2)$ where $n > 5$

For $1 \leq i \leq n-1$:

$f(v_ie_i) = 2i - 1$; $\quad f(e_iv_{i+1}) = 2i$;

For $1 \leq i \leq n-4$:

$f(e_ie_{i+1}) = 2(n-1) + i$;

$f(e_{n-3}e_{n-2}) = 3n - 4$; $\quad f(e_{n-2}e_{n-1}) = 3n - 5$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $M(P_n)$ is antimagic.

Illustration 2.2. Middle graph of path P_5 and its antimagic labeling is shown in *Figure* 1.

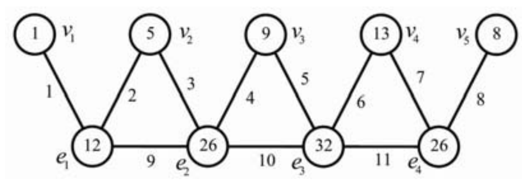

Figure 1

Theorem 2.3. Middle graph of cycle C_n is antimagic.
Proof. Let $v_1, v_2, \dots, v_n$ be the vertices and $e_1, e_2, \dots, e_n$ be the edges of cycle C_n and $G = M(C_n)$ be the middle graph of cycle C_n. According to the definition of middle graph $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = \{v_i e_i ; 1 \le i \le n, v_i e_{i-1} ; 2 \le i \le n, e_i e_{i+1} ; 1 \le i \le n-1, e_n e_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$. To define $f : E(G) \to \{1, 2, \dots 3n\}$, we consider following two cases.

Case 1: $n \equiv 1\ (mod\ 2)$
For $1 \le i \le n$:
$f(v_ie_i) = 2i$;
For $1 \le i \le n-1$:
$f(e_iv_{i+1}) = 2i + 1$; $f(e_nv_1) = 1$;
$f(e_ie_{i+1}) = 2n + 1 + i$; $f(e_ne_1) = 2n + 1$;

Case 2: $n \equiv 0\ (mod\ 2)$
$f(v_1e_1) = 2$;
For $2 \le i \le n$:
$f(v_ie_i) = 2i$;
For $1 \le i \le n-1$:
$f(e_iv_{i+1}) = 2i + 1$; $f(e_nv_1) = 1$;
For $1 \le i \le n-1$:
$f(e_ie_{i+1}) = 2n + 1 + i$; $f(e_ne_1) = 2n + 1$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $M(C_n)$ is antimagic.

Illustration 2.4. Middle graph of cycle C_5 and its antimagic labeling is shown in *Figure* 2.

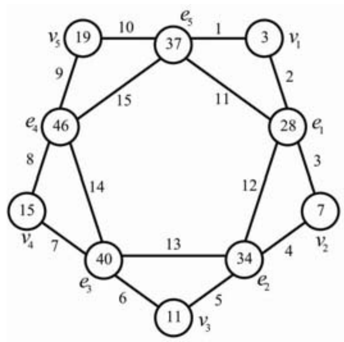

Figure 2

Therom 2.5. Total graph of path P_n is antimagic.
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n and $G = T(P_n)$ be the total graph of path P_n with $V(T(P_n)) = V(P_n) \cup E(P_n)$ and $E(T(P_n)) = \{v_i v_{i+1}; 1 \leq i \leq n-1, v_i e_i; 1 \leq i \leq n-1, e_i e_{i+1}; 1 \leq i \leq n-2, v_i e_{i-1}; 2 \leq i \leq n\}$. Here $|V(G)| = 2n-1$ and $|E(G)| = 4n-5$. Define $f: E(G) \rightarrow \{1, 2, \ldots 4n-5\}$ as follows.

<u>For $1 \leq i \leq n-2$:</u>
$f(v_i v_{i+1}) = 4i$; $f(v_{n-1} v_n) = 4n-5$;
$f(e_i e_{i+1}) = 4i-3$;
<u>For $1 \leq i \leq n-2$:</u>
$f(v_i e_i) = 4i-1$;
<u>For $2 \leq i \leq n-1$:</u>
$f(v_i e_{i-1}) = 4i-2$; $f(v_n e_{n-1}) = 4n-7$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $T(P_n)$ is antimagic.

Illustration 2.6. Total graph of path $T(P_6)$ and its antimagic labeling is shown in *Figure* 3.

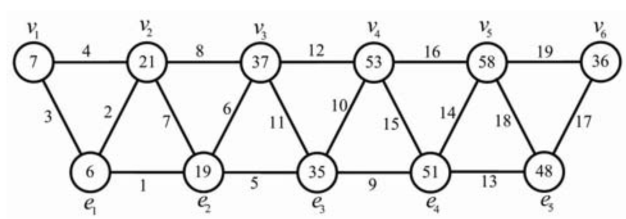

Figure 3

Theorem 2.7. Total graph of cycle C_n is antimagic.
Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n and $G=T(C_n)$ be the total graph of cycle C_n with $V(T(C_n))=V(C_n) \cup E(C_n)$ and $E(T(C_n)) = \{v_i v_{i+1}; 1 \leq i \leq n-1, v_n v_1, v_i e_i; 1 \leq i \leq n, e_i e_{i+1}; 1 \leq i \leq n-1, e_n e_1, v_i e_{i+1}; 2 \leq i \leq n, v_1 e_n\}$. Here $|V(G)| = 2n$ and $|E(G)| = 4n$. To define $f: E(G) \rightarrow \{1, 2, \ldots 4n\}$ we consider following two cases.

Case 1: $n \equiv 1 \ (mod\ \ 2)$

For $1 \le i \le n-1$:

$f(v_i v_{i+1}) = 3n+i$; $\quad f(v_n v_1) = 4n$;

For $2 \le i \le n-1$:

$f(e_i e_{i+1}) = i+1$; $\quad f(e_n e_1) = 1$;

For $1 \le i \le n$:

$f(v_i e_i) = n+2i$;

For $2 \le i \le n$:

$f(v_i e_{i-1}) = n-1+2i$; $\quad f(v_1 e_n) = n+1$;

Case 2: $n \equiv 0 \ (mod\ \ 2)$

Sub case 1: $n \equiv 0 \ (mod\ \ 4)$

For $1 \le i \le n-1$:

$f(v_i v_{i+1}) = 3n+i$; $\quad f(v_n v_1) = 4n$;

For $2 \le i \le n-1$:

$f(e_i e_{i+1}) = i+1$; $\quad f(e_n e_1) = 2$;

$f(e_1 e_2) = 1$;

For $1 \le i \le n$:

$f(v_i e_i) = n+2i$;

For $2 \le i \le n$:

$f(v_i e_{i-1}) = n-1+2i$; $\quad f(v_1 e_n) = n+1$;

Sub case 2: $n \equiv 2 \ (mod\ \ 4)$

$f(v_1 v_2) = 3n+2$; $\quad f(v_2 v_3) = 3n+1$;

For $3 \le i \le n-1$:

$f(v_i v_{i+1}) = 3n+i$;

For $2 \le i \le n-1$:

$f(e_i e_{i+1}) = i+1$;

$f(e_n e_1) = 2$; $\quad f(e_1 e_2) = 1$;

For $1 \le i \le n$:

$f(v_i e_i) = n+2i$;

For $2 \le i \le n$:

$f(v_i e_{i-1}) = n-1+2i$; $\quad f(v_1 e_n) = n+1$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $T(C_n)$ is antimagic.

Illustration 2.8. Total graph of cycle C_6 and its antimagic labeling is shown in *Figure* 4.

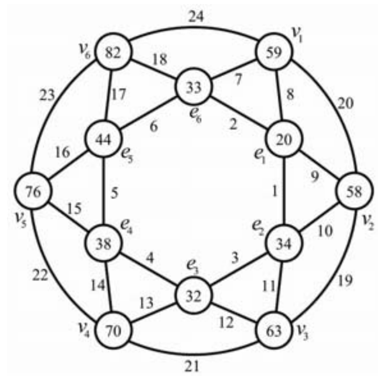

Figure 4

Theorem 2.9. Splitting graph of path P_n is antimagic.

Proof. Let v_1, v_2, ... , v_n be the vertices and e_1, e_2, ... , e_{n-1} be the edges of path P_n. Let $v'_1, v'_2, \ldots, v'_n$ be the newly added vertices to form the splitting graph of path P_n. Let G= $S(P_n)$ be the splitting graph of path P_n . $V(S'(P_n))$ = $\{v_i, v'_i / 1 \le i \le n\}$ and $E(S'(P_n))$ $= \{v'_i v_{i+1}; 1 \le i \le n-1, v'_i v_{i-1}; 2 \le i \le n, v_i v_{i+1}; 1 \le i \le n-1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n - 3$. Define $f : E(G) \rightarrow \{1, 2, \ldots , 3n - 3\}$ as follows.

For $1 \le i \le n-1$:

$f\left(v_i v_{i+1}\right) \ = \ 3i$;

$f(v'_i v_{i+1}) = 3i - 2$;

$f(v_i v'_{i+1}) = 3i - 1$;

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $S'(P_n)$ is antimagic.

Illustration 2.10. Splitting graph of path P_6 and its antimagic labeling is shown in *Figure* 5.

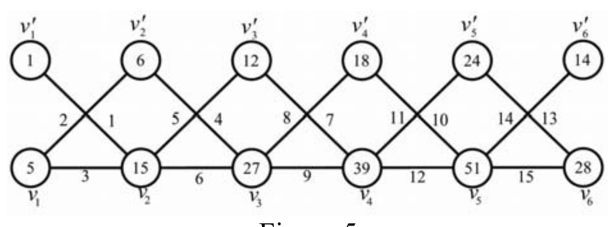

Figure 5

Theorem 2.11. Splitting graph of cycle C_n is antimagic.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n. Let $v'_1, v'_2, \ldots, v'_n$ be the newly added vertices to form the splitting graph of cycle C_n. Let G=$S'(C_n)$ be the splitting graph of cycle C_n. $V(S'(C_n))$ = $\{v_i, v'_i / 1 \le i \le n\}$ and $E(S'(C_n))$

$=\{v_i'v_{i+1}; 1 \le i \le n-1, v_n'v_1, v_1'v_n, v_i'v_{i-1}; 2 \le i \le n, v_iv_{i+1}; 1 \le i \le n-1, v_nv_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$. To define $f: E(G) \to \{1, 2, \dots, 3n\}$ we consider following two cases.

Case 1: $n \equiv 1 (mod\, 2)$

For $1 \le i \le n-1$:

$f(v_iv_{i+1}) = 2n+1+i$; $\quad f(v_nv_1) = 2n$;

$f(v_i'v_{i+1}) = \begin{cases} 2i+1, & i \equiv 1(\bmod 2); \\ 2i, & otherwise. \end{cases}$ $\quad f(v_n'v_1) = 2n+1$;

For $2 \le i \le n$:

$f(v_i'v_{i-1}) = \begin{cases} 2i-1, & i \equiv 1(\bmod 2); \\ 2i-2, & otherwise. \end{cases}$ $\quad f(v_1'v_n) = 1$;

Case 2: $n \equiv 0$ $(mod\ \ 2)$

For $1 \le i \le$ n – 1 :

$f(v_iv_{i+1}) = 2n+1+i$; $\quad f(v_nv_1) = 3n$;

Sub Case 1: $n \equiv 0$ $(mod\ \ 4)$, $n \ne 4$

$f(v_2'v_1) = 4$; $\quad f(v_2'v_3) = 2$;

For $1 \le i \le n$, (i $\ne$ 2):

$f(v_i'v_{i+1}) = \begin{cases} 2i+1, & i \equiv 1(\bmod 2); \\ 2i, & otherwise. \end{cases}$ $\quad f(v_1'v_n) = 1$;

For $3 \le i \le n$:

$f(v_i'v_{i-1}) = \begin{cases} 2i-1, & i \equiv 1(\bmod 2); \\ 2i-2, & otherwise. \end{cases}$

Sub Case 2: $n = 4$ and $n \equiv 1$ $(mod\ \ 4)$

$f(v_1'v_n) = 3$; $\quad f(v_1'v_2) = 1$;

For $2 \le i \le n$:

$f(v_i'v_{i+1}) = \begin{cases} 2i+1, & i \equiv 1(\bmod 2); \\ 2i, & otherwise. \end{cases}$ $\quad f(v_i'v_{i-1}) = \begin{cases} 2i-1, & i \equiv 1(\bmod 2); \\ 2i-2, & otherwise. \end{cases}$

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $S'(C_n)$ is antimagic.

Illustration 2.12. Splitting graph of cycle C_4 and its antimagic labeling is shown in *Figure* 6.

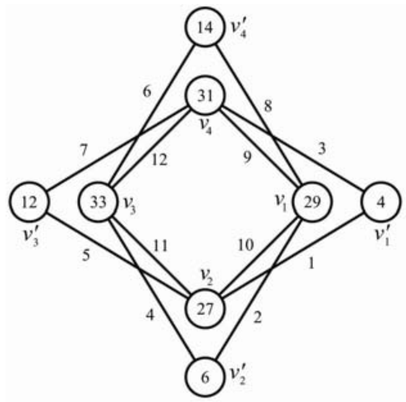

Figure 6

Theorem 2.13. Shadow graph of path P_n is antimagic.

Proof. Let P_n', P_n'' be two copies of path P_n. We denote the vertices of first copy of P_n by $v_1', v_2', ..., v_n'$ and second copy by $v_1'', v_2'', ..., v_n''$. Let G be $D_2(P_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n - 4$. Define $f : E(G) \rightarrow \{1, 2, \dots, 4n - 4\}$ as follows.

<u>For $1 \le i \le n - 1$:</u>

$f(v_i'v_{i+1}') = 4i$; $f(v_i''v_{i+1}'') = 4i - 3$; $f(v_i'v_{i+1}'') = 4i - 1$;

<u>For $2 \le i \le n$:</u>

$f(v_i'v_{i-1}'') = 4i - 2$;

Above defined edge labeling function will generate all distinct vertex labels satisfying the condition for antimagic labeling. Hence $D_2(P_n)$ is antimagic.

Illustration 2.14. Shadow graph of path P_6 and its antimagic labeling is shown in Figure 7.

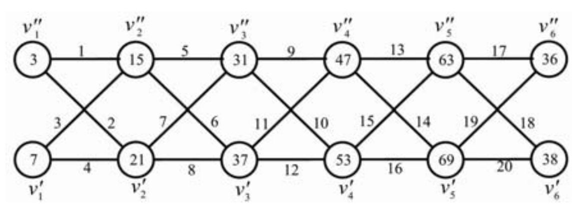

Figure 7

Theorem 2.15. Shadow graph of cycle C_n is antimagic.

Proof. Let C_n', C_n'' be two copies of cycle C_n. We denote the vertices of first copy of C_n by $v_1', v_2', \dots, v_n'$ and second copy by $v_1'', v_2'', ..., v_n''$. Let G be $D_2(C_n)$ with $|V(G)| = 2n$ and $|E(G)| = 4n$. To define $f : E(G) \rightarrow \{1, 2, \dots, 4n\}$ we consider following three cases.

Case 1: $n \equiv 1 (mod\ 2)$

For $1 \leq i \leq n-1$:

$f(v''_i v''_{i+1}) = i$; $f(v''_n v''_1) = n$;

$f(v'_i v'_{i+1}) = 3n + i$; $f(v'_n v'_1) = 4n$;

$f(v''_i v'_{i+1}) = n + 2i - 1$; $f(v''_n v'_1) = 3n - 1$;

$f(v'_i v''_{i-1}) = n + 2i$; $f(v'_n v''_1) = 3n$;

Case 2: $n \equiv 0(mod\ 2), n \neq 6$

$f(v''_n v''_1) = 1$; $f(v'_n v'_1) = 4n$;

$f(v'_n v''_1) = n + 1$; $f(v'_1 v''_n) = 3n$;

For $1 \leq i \leq n-1$:

$f(v''_i v''_{i+1}) = 3n + i$; $f(v'_i v'_{i+1}) = i + 1$;

$f(v'_i v''_{i+1}) = n + 1 + 2i$;

For $2 \leq i \leq n$:

$f(v'_i v''_{i-1}) = n + 2i$;

Case 3: For $n = 6$, antimagic labeling of $D_2(C_6)$ is shown in below *Figure* 8.

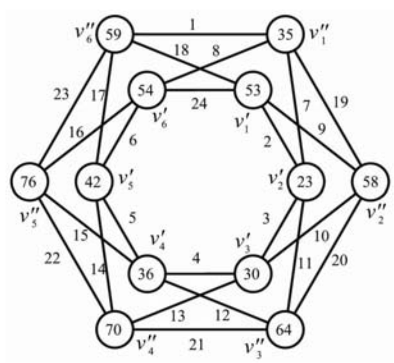

Figure 8

Above defined edge labeling function will generate all the distinct vertex labels satisfying the condition for antimagic labeling. Hence $D_2(C_n)$ is antimagic.

Illustration 2.16. Shadow graph of cycle C_5 and its antimagic labeling is shown in Figure 9.

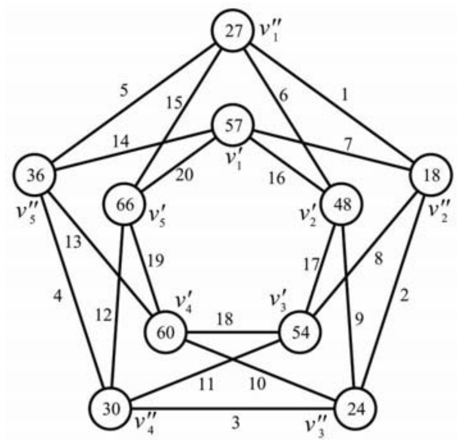

Figure 9

3. Concluding Remarks

We have investigated antimagic labeling for shadow graph, middle graph and total graph of P_n and C_n. More exploration is possible for other graph families and in the context of different graph labeling problems.

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