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SOME MORE RESULTS ON EVEN MEAN LABELING

S. K. Vaidya [a,*], N. B. Vyas [b]

[a] Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India
[b] Atmiya Institute of Technology and Science, Rajkot - 360005, Gujarat, India

∗ **Corresponding author: S. K. Vaidya** Tel.: +91-9825292539; e-mail: samirkvaidya@yahoo.co.in
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ABSTRACT

A function f: V(G) →{2,4, ..., 2|E(G)|} is called an even mean labeling of graph G if it is injective and when each edge e=uv is labeled with average sum of f(u) and f(v) then the resulting edge labels are distinct. Here we investigate even mean labeling for the graphs obtained by some graph operations on standard graphs.

Keywords: *Even Mean labeling; Middle graph; Splitting graph; Triangular Snake and Ladder.*
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1. INTRODUCTION

Throughout this work, by a graph we mean finite, connected, undirected, simple graph $G = (V(G),E(G))$ of order $|V(G)|$ and size $|E(G)|$.

A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a *vertex labeling* (an *edge labeling*).

According to Beineke and Hegde[1], a labeling of discrete structure is a frontier between graph theory and theory of numbers. A latest survey on various graph labeling problems can be found in Gallian[2].

The function f is called *mean labeling* of graph G if $f:V(G) \rightarrow \{0, 1, 2, \dots , |E(G)|\}$ is injective and the induced function $f^*: E(G) \rightarrow \{1, 2, \dots , |E(G)|\}$ defined as

$$f^*(e=uv) = \begin{cases} \dfrac{f(u)+f(v)}{2}, & if\ f(u)+f(v) \text{ is even;} \\ \dfrac{f(u)+f(v)+1}{2}, & if\ f(u)+f(v) \text{ is odd.} \end{cases}$$

is bijective. A graph which admits mean labeling is called a *mean graph*.

The mean labeling was introduced by Somasundaram and Ponraj [3] and they proved that the graphs P_n, C_n, $P_n \times P_m$, $P_m \times C_n$ are mean graphs. The graphs $P_m[P_2]$, P_n^2, $M(P_n)$ and some cycle related graphs are proved to be mean graphs by Vaidya and Lekha [4].

According to Pricilla [5], a function $f: V(G) \rightarrow \{2,4, \ldots, 2|E(G)|\}$ is called an *even mean labeling* of graph G if it is injective and when each edge $e = uv$ is labeled with the average sum of $f(u)$ and $f(v)$ then the resulting edge labels are distinct.

Vaidya and Vyas [6] have proved that shadow graph, middle graph, total graph, splitting graph and cube graph of path admit even mean labeling. They also proved that square graph and splitting graph of bistar and double fan are even mean graphs.

The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are incident edges of G or one is a vertex of G and the other is an edge incident on it. For a graph G the *splitting graph* $S'(G)$ is obtained by adding new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' respectively. Duplication of a vertex v_k by a new edge $e = v_k' v_k''$ in a graph G produces a new graph G' such that $N(v_k') \cap N(v_k'') = v_k$. The *Triangular snake* T_n is obtained from path P_n by joining vertices v_i and v_{i+1} by v_i' where $1 \leq i \leq n-1$. The *Quadrilateral snake* QS_n is obtained from path P_n by joining v_i to v_i' and v_{i+1} to v_i'' where v_i' and v_i'' are the end points of path P_2 and $1 \leq i \leq n-1$. The *Ladder graph* is obtained by $P_n \times P_2$ while *Circular ladder* is obtained by $C_n \times P_2$. For any undefined term in graph theory, we rely upon West [7].

2. MAIN RESULTS

Theorem 2.1. $M(C_n)$ is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n and $G = M(C_n)$ be the middle graph of cycle C_n. According to the definition of middle graph, $V(M(C_n)) = V(C_n) \cup E(C_n)$ and $E(M(C_n)) = \{v_i e_i; 1 \leq i \leq n, v_i e_{i-1}; 2 \leq i \leq n, e_i e_{i+1}; 1 \leq i \leq n-1, e_n e_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f: V(G) \rightarrow \{2, 4, \ldots, 2|E(G)|\}$ as follows.

Case 1: n is even

<u>For $1 \leq i \leq \frac{n}{2}$:</u>

$f(v_i) = 4i - 2, i \neq \frac{n}{2}$;

$f(v_{\frac{n}{2}}) = 4n - 2$;

$f(v_i') = 4i$;

<u>For $\frac{n}{2} < i \leq n$:</u>

$f(v_i) = 2n + 4i - 2;$

$f(v_i') = 2n + 4i;$

Case 2: n is odd

For $1 \le i \le \frac{n+1}{2}$:

$f(v_i) = 4i - 2;$

$f(v_i') = 4i;$

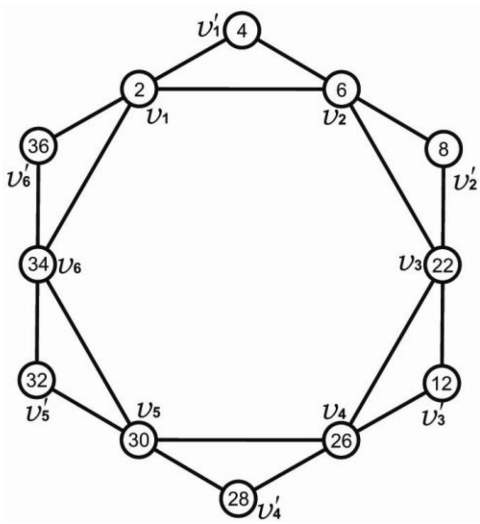
For $\frac{n+1}{2} < i \le n$:

$f(v_i) = 2n + 4i - 2;$

$f(v_i') = 2n + 4i;$

The above defined function *f* provides an even mean labeling for middle graph of cycle.
That is, $M(C_n)$ is an even mean graph. □

Illustration 2.2. $M(C_6)$ and its even mean labeling is shown in *Figure* 1.



Theorem 2.3. The graph obtained by duplicating each vertex by an edge in cycle C_n is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n. Let G be the graph obtained by duplicating each vertex v_i of C_n by an edge $v_i'v_i''$, where $1 \le i \le n$. Here $|V(G)| = 3n$ and $|E(G)| = 4n$. We define $f: V(G) \to \{2, 4, \ldots, 2|E(G)|\}$ as follows.

Case 1: n is even

$f(v_1) = 2;$

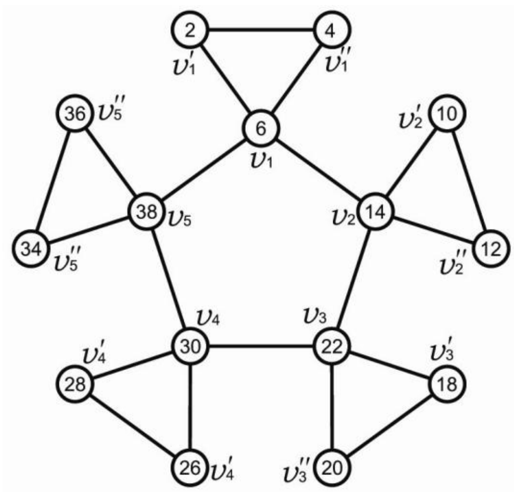
$f(v_1') = 4;$
$f(v_1'') = 6;$
<u>For $2 \le i \le n$:</u>
$f(v_i) = 8i - 2;$
$f(v_i') = 8i - 6;$
$f(v_i'') = 8i - 4;$

Case 2: n is odd

$f(v_1) = 6;$
$f(v_1') = 2;$
$f(v_1'') = 4;$
<u>For $2 \le i \le n$:</u>
$f(v_i) = 8i - 2;$
$f(v_i') = 8i - 6;$
$f(v_i'') = 8i - 4;$

The above defined function *f* provides an even mean labeling for *G*.
Hence, *G* is an even mean graph. □

Illustration 2.4. The graph obtained by duplicating each vertex by an edge in cycle C_5 and its even mean labeling is shown in *Figure* 2.



Theorem 2.5. $S'(C_n)$ is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n. Let $v_1', v_2', \ldots, v_n'$ be the newly added vertices to form the splitting graph of cycle C_n. Let $G = S'(C_n)$ be the splitting graph of cycle C_n. Then $V(S'(C_n)) = \{v_i, v_i' / 1 \le i \le n\}$ and $E(S'(C_n)) = \{v_i'v_{i+1}; 1 \le i \le n-1, v_n'v_1,$

$v_1'v_n, v_i'v_{i-1}; 2 \le i \le n, v_iv_{i+1}; 1 \le i \le n-1, v_nv_1\}$. Here $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f: V(G) \to \{2, 4, \ldots, 2|E(G)|\}$ as follows.

For $1 \le i \le n$:

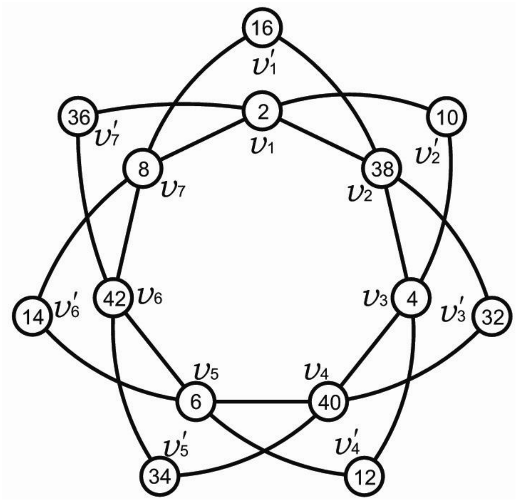
$$f(v_i) = \begin{cases} \dfrac{i+1}{2}, & \text{if } i \equiv 1(\bmod 2); \\ \left(\dfrac{11n-1}{2}\right) + (i-2), & \text{otherwise.} \end{cases}$$

$$f(v_1') = 2(n+1);$$

$$f(v_i') = \begin{cases} n+1+i, & \text{if } i \equiv 0(\bmod 2); \\ \left(\dfrac{9n+1}{2}\right) + (i-3), & \text{otherwise}(i \ne 1). \end{cases}$$

The above defined function *f* provides an even mean labeling for *G*.
Hence, *G* is an even mean graph. □

Illustration 2.6. $S'(C_7)$ and its even mean labeling is shown in *Figure* 3.



Theorem 2.7. T_n is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. To construct triangular snake T_n from path P_n, join v_i and v_{i+1} by v_i' where $1 \le i \le n-1$. Here $|V(T_n)| = 2n-1$ and $|E(T_n)| = 3(n-1)$. We define $f: V(T_n) \to \{2, 4, \ldots, 2|E(T_n)|\}$ as follows.

For $1 \le i \le n$:

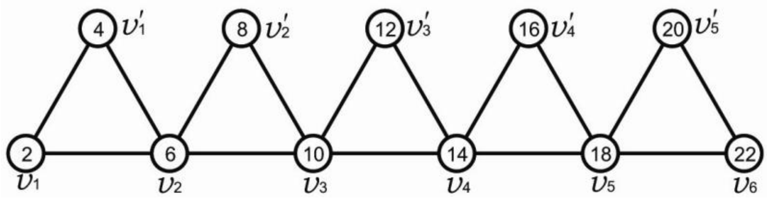
$$f(v_i) = 4i - 2;$$

For $1 \le i \le n-1$:

$f(v_i') = 4i;$

The above defined function *f* provides an even mean labeling for *G*.
Hence, T_n is an even mean graph. □

Illustration 2.8. T_6 and its even mean labeling is shown in *Figure* 4.



Theorem 2.9. QS_n is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. To construct Quadrilateral snake QS_n from path P_n join v_i to v_i' and v_{i+1} to v_i'' where v_i' and v_i'' are the end points of P_2 and $1 \le i \le n-1$. Here $|V(QS_n)| = 3n-2$ and $|E(QS_n)| = 4(n-1)$. We define $f : V(QS_n) \to \{2, 4, \ldots, 2|E(QS_n)|\}$ as follows.

For $1 \le i \le n$:

$f(v_i) = 2i;$

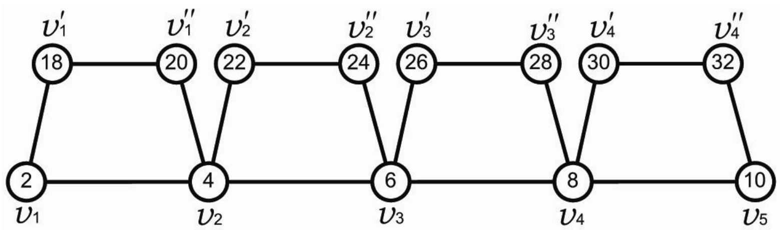
For $1 \le i \le n-1$:

$f(v_i') = 4n - 6 + 4i;$

$f(v_i'') = 4(n-1) + 4i;$

The above defined function *f* provides an even mean labeling for graph QS_n.
Hence, QS_n is an even mean graph. □

Illustration 2.10. QS_5 and its even mean labeling is shown in *Figure* 5.



Theorem 2.11. DT_n is an even mean graph.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n. To construct double triangular snake DT_n from path P_n, join v_i and v_{i+1} to two new vertices v_i' and v_i'' where $1 \le i \le n-1$

. Here $|V(DT_n)|=3n-2$ and $|E(DT_n)|=5(n-1)$. We define $f:V(DT_n)\rightarrow\{2,4,\ldots,2|E(DT_n)|\}$ as follows.

For $1\leq i\leq n$:

$f(v_i)=6i-4;$

For $1\leq i\leq n-1$:

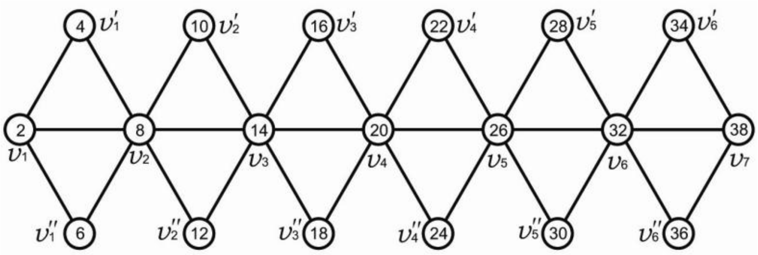
$f(v_i')=6i-2;$

$f(v_i'')=6i;$

The above defined function *f* provides an even mean labeling for DT_n.

Hence, DT_n is an even mean graph. □

Illustration 2.12. DT_7 and its even mean labeling is shown in *Figure* 6.



Theorem 2.13. $D(QS_n)$ is an even mean graph.

Proof. Let $v_1,v_2,\ldots,v_n$ be the vertices and $e_1,e_2,\ldots,e_{n-1}$ be the edges of path P_n. To construct Double Quadrilateral snake $D(QS_n)$ from path P_n, join v_i to v_i' and u_i' while join v_{i+1} to v_i'' and u_i'' where v_i' is adjacent to v_i'' , u_i' is adjacent to u_i'' and $1\leq i\leq n-1$. Here $|V(D(QS_n))|=5n-4$ and $|E(D(QS_n))|=7(n-1)$. We define $f:V(D(QS_n))\rightarrow\{2,4,\ldots,2|E(D(QS_n))|\}$ as follows.

For $1\leq i\leq n$:

$f(v_i)=10i-8;$

For $1\leq i\leq n-1$:

$f(v_i')=10i-6;$

$f(v_i'')=10i-2;$

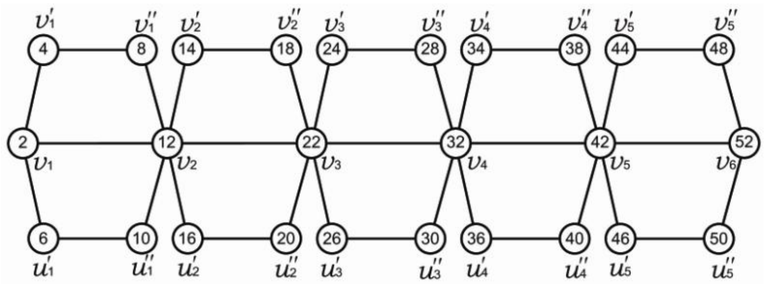
$f(u_i')=10i-4;$

$f(u_i'')=10i;$

The above defined function *f* provides an even mean labeling for $D(QS_n)$.

Hence, $D(QS_n)$ is an even mean graph. □

Illustration 2.14. $D(QS_6)$ and its even mean labeling is shown in *Figure* 7.



Theorem 2.15. Ladder is an even mean graph.

Proof. Let G be the graph $P_n \times P_2$ and $V(G) = \{v_{ij} / i = 1, 2, \ldots, n$ and $j = 1, 2\}$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n - 2$. We define $f : V(G) \to \{2, 4, \ldots, 2|E(G)|\}$ as follows.

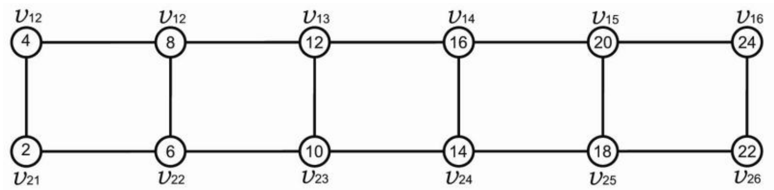
$\underline{\text{For } 1 \le i \le n:}$

$f(v_{1i}) = 4i;$

$f(v_{2i}) = 4i - 2;$

The above defined function *f* provides an even mean labeling for G.
Hence, G is an even mean graph. □

Illustration 2.16. $P_6 \times P_2$ and its even mean labeling is shown in *Figure* 8.



Theorem 2.17. Circular ladder is an even mean graph.

Proof. Let G be the graph $C_n \times P_2$ and $V(G) = \{v_{ij} / i = 1, 2, \ldots, n$ and $j = 1, 2\}$. We note that $|V(G)| = 2n$ and $|E(G)| = 3n$. We define $f : V(G) \to \{2, 4, \ldots, 2|E(G)|\}$ as follows.

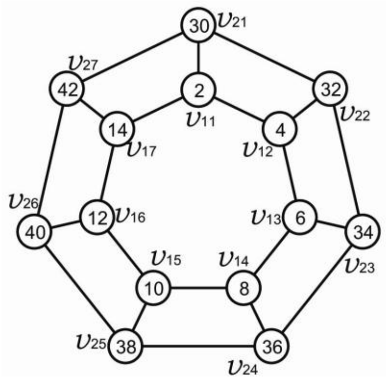
$\underline{\text{For } 1 \le i \le n:}$

$f(v_{1i}) = 2i;$

$f(v_{2i}) = 4n + 2i;$

The above defined function *f* provides an even mean labeling for G.
Hence, G is an even mean graph. □

Illustration 2.18. $C_7 \times P_2$ and its even mean labeling is shown in *Figure* 9.



CONCLUDING REMARKS

It is always interesting to find out graph which admit a particular labeling. Here we investigate even mean labeling of some cycle related graphs, triangular snake, quadrilateral snake, double triangular snake, double quadrilateral snake, ladder and circular ladder. To investigate similar results for other graph families is an open area of research.

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