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BEST PROXIMITY POINTS FOR GF PROXIMAL CONTRACTION AND ITS APPLICATIONS TO VARIATIONAL INEQUALITY PROBLEM

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Abstract: The main concept of this paper is to introduce the notion of GF and GF generalized proximal contraction and to establish the existence and uniqueness of best proximity point for nonself mapping satisfying these contractions in complete metric space. An example is also supplied in support of our result. As application part, we introduce variational inequality problem.

1. INTRODUCTION

In 1922, Banach [15] introduced Banach's contraction principle which states that, "Let T be a self mapping on X where (X, d) is a complete metric space such that $d(Tx, Ty) \leq \alpha\, d(x, y)$ for all $x, y \in X$, where $\alpha \in [0,1)$ then T has a unique fixed point."

Mathematically, if A and B are two non-empty subsets of metric space (X, d) and $T: A \rightarrow B$ be nonself mapping. Then for the existence of fixed point of mapping T, we have a necessary but not sufficient condition, which is $T(A) \cap A \neq \phi$. If $T(A) \cap A = \phi$, then $d(x, Tx) > 0$ for all $x \in A$ that is, set of fixed points is empty. In such case, researcher try to established an element $x \in A$ which is in some sense closest to Tx. If such an element exists, we call it best proximity point, that is , an element $x \in A$ is called best proximity point of T if

$$d(x, Tx) = d(A, B) \text{ where } d(A, B) = \inf\{d(x, y): x \in A, y \in B\}. \quad (1)$$

Best proximity point theory developed in this direction. The main motive of best proximity point theory is to establish sufficient condition which gives us the assurance of existence of best proximity points.

Firstly, the notion of best proximity point was initiated by Fan [1] for normed spaces which states that

Theorem 1.1[1] Let A be a nonempty compact convex subset of a normed linear space X and $T: A \rightarrow X$ be a continuous mapping. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A)$.

Afterwards, several mathematicians make practical and effective use of the Fan's theorem in many direction.

Inspiring from[2,3,7] we prove the result of Roger- Hardy to nonself mapping by introducing GF proximal contraction.

In 2012, Wardowski [6] introduced a new type of contractions which is known as F-contraction as follow:

Definition 1.2[6] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $x, y \in \mathbb{R}^+$ such that, if $x < y$, then $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We represent ΔF, the family of all functions F that satisfy the conditions (F1)–(F3).

Definition 1.3[6] Let (X, d) be a metric space. Then $T : X \to X$ is called F-contraction if there exists $T > 0$ such that

$$d(T x, T y) > 0 \Rightarrow \tau + F(d(T x, T y)) \leq F(d(x, y)), \tag{2}$$

for all $x, y \in X$, where $F \in \Delta F$.

When we consider in (2) the different types of the mapping *F* then we obtain the variety of contractions, some of them are of a type known in the literature. See the following examples

Example 1.4[6] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that *F* satisfies (F1)-(F3) ((F3) for any $k \in (0, 1)$). Each mapping

$T:X \to X$ satisfying (2) is an *F*-contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{for all } x, y \in X, Tx \neq Ty. \tag{3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality

$$d(Tx, Ty) \leq e^{-\tau} d(x, y)$$

also holds, i.e. *T* is a Banach contraction [15].

Hussain et al.[8] introduced a family of functions as follows:

Definition 1.5[8] Let ΔG denotes the set of all functions $G: R^{+4} \to R^+$ satisfying:

(G) for all $t_1, t_2, t_3, t_4 \in R^+$ with $t_1t_2t_3t_4 = 0$ there exists $\tau > 0$ such that

$G(t_1, t_2, t_3, t_4) = \tau$.

Example 1.6[8] If $G(t_1, t_2, t_3, t_4) = L \min\{t_1, t_2, t_3, t_4\} + \tau$ where $L \in R+$ and $\tau > 0$, then $G \in \Delta G$.

Example 1.7[8] If $G(t_1, t_2, t_3, t_4) = L \ln(\min\{t_1, t_2, t_3, t_4\} + 1) + \tau$ where $L \in R^+$ and $\tau > 0$, then $G \in \square G$.

To start our main results, firstly we focus on some most basic definitions and results.

Let R, N and Q denote the set of all real numbers, positive integers and rational numbers, respectively. Let A and B be two nonempty subsets of a metric space (X, d). By using the usual notation in nonlinear analysis, we recall the following notions:

$$A_0 = \{x \in A : d(x, y) = d(A,B), \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A,B), \text{ for some } x \in A\}.$$

As A_0 and B_0 are nonempty sets given by Kier et al.[9]. Also, A_0 is contained in the boundary of A, proved by Sadiq Bashan ad Veermani[10]. In the same pattern, we try to establish new types of GF and GF generalized proximal contraction and for this purpose we recollect the fundamental definitions, as follows:

Definition 1.8[14] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have Property if and only if $d(u, Tx) = d(A, B)$, $d(v, Ty) = d(A, B)$,

$$\Rightarrow d(u, v)=d(Tx, Ty), \tag{4}$$

where $u, v \in A_0$ and $Tx, Ty \in B_0$.

Definition 1.9[11] Let A and B be two nonempty subsets of a metric space (X, d). A non-self mapping $T: A \to B$ is said to be a proximal contraction of the first kind if

$$d(u, Tx) = d(A, B),\ d(v, Ty) = d(A, B) \Rightarrow d(u, v) \leq k\, d(x, y),$$

for all $u, v, x, y \in A$, where $k \in [0,1)$.

Definition 1.10[11] Let A and B be two nonempty subsets of a metric space (X, d). A non-self mapping $T: A \to B$ is said to be a proximal contraction of the second kind if

$d(u, Tx) = d(A, B)= d(v, Ty) \Rightarrow d(Tu, Tv) \leq k\, d(Tx, Ty)$, for all $u, v, x, y \in A$, where $k \in [0,1)$.

Many authors generalized these concepts of proximal and proved their best approximation theorems.

Afterthat Basha [4] introduced the concept of generalized proximal contraction of first and second kind which are defined below:

Definition1.11[4] A mapping $T : A \to B$ is said to be generalized proximal contraction of the first kind if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$such that the conditions

$d(u, Tx)=d(A, B)$ and $d(v, Ty)=d(A, B)$

imply the inequality that

$d(u, v)\leq \alpha d(x, y) + \beta d(x, u) + \gamma d(y, v) + \delta[d(x, v) + d(y, u)]$ for all u, v, x, y in A.

If T is a self-mapping on A, then the requirement in the preceding definition reduces to the condition that

$d(Tx,Ty)\leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta[d(x, Ty) + d(y, Tx)]$.

Definition 1.12[4] A mapping $T : A \to B$ is said to be a generalized proximal contraction of the second kind if there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that the conditions $d(u, Tx)=d(A, B)$ and $d(v, Ty)=d(A, B)$

imply the inequality

$d(Tu, Tv)\leq\alpha d(Tx, Ty) + \beta d(Tx, Tu) + \gamma d(Ty, Tv) + \delta[d(Tx, Tv)+d(Ty, Tu)]$ for all u, v, x, y in A.

Definition 1.13[7] Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is called a Roger–Hardy type generalized F-contraction mapping if there exists $\tau > 0$ and $F \in \square F$ such that

$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, T x) + \gamma d(y, Ty) + \eta d(x, Ty) + \delta d(y, T x))$ for all $x, y \in X$ and $\alpha, \beta, \gamma, \eta, \delta \geq 0$ with $\alpha + \beta + \gamma + \eta + \delta < 1$.

2. GF PROXIMAL CONTRACTIONS

In this section, we introduce GF proximal contraction and prove the existence of unique best proximity point for GF proximal contraction in complete metric space.

Definition 2.1 Let (X, d) be a complete metric space and T be a non self-mapping from A to B where A and B are closed subset of complete metric space (X, d). T is said to be an GF- proximal contraction if for $x, y \in X$ with $d(Tx, T y) > 0$ we have,

$G(d(x, u), d(y, v), d(x, v), d(y, u)) + F\, d(u, v) \leq F\, d(x, y)$

where $G \in \Delta G$ and $F \in \Delta F$.

Theorem 2.2 Let (X, d) be complete metric space and A, B be two non empty closed subset of X. A_0, B_0 be nonempty subset of A and B respectively . Let T:

$A \to B$ be a mapping satisfying following assertions:

I. T is continuous;

II. T has P-property;

III. T is an GF-proximal contraction ;

IV. $T(A_0) \subset B_0$.

Then T has unique element $x \in A$ such that $d(x, Tx)=d(A, B)$ and if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1}, Tx_n)=d(A, B)$ for all $n \geq 0$, then $\lim_{n\to\infty} x_n = x$.

Proof Since A_0 is nonempty, we take $x_0 \in A$. As $Tx_0 \in T(A_0) \subset B_0$, we can find $x_1 \in A_0$ such that $d(x_1, T x_0) = d(A, B)$.

Similarly, since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that

$d(x_2, T x_1) = d(A, B)$.

Repeating this process, we can get a sequence $\{x_n\}$ in A_0 satisfying

$d(x_{n+1}, Tx_n) = d(A, B)$ for any $n \in N$.

Since (A, B) has the P-property, we have that

$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$ for any $n \in N$.

As T is GF-proximal contraction, for any $n \in N$, we have that

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}),d(x_{n-1}, x_{n+1}),d(x_n, x_n)) + Fd(x_n, x_{n+1})d\leq Fd(x_{n-1}, x_n)$$

Now since, $d(x_{n-1}, x_n).d(x_n, x_{n+1}).d(x_{n-1}, x_{n+1}).0 = 0$, so from (G) there exists $\tau > 0$ such that,

$$Gd(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0 = \tau.$$

we deduce that,

$$Fd(x_n, x_{n+1}) \leq Fd(x_{n-1}, x_n) - \tau.$$

$$\leq F\, d(x_{n-2}, x_{n-1}) - 2\tau \leq \ldots \leq F(d(x_0, x_1)) - n\tau.$$

This implies that

$$Fd(x_n, x_{n+1}) \leq F(d(x_0, x_1)) - n\tau.$$

By taking limit as $n \to \infty$ in we have,

$\lim_{n\to\infty} F\, d(x_n, x_{n+1}) = -\infty$, and since, $F \in \Delta F$ we obtain,

$$\lim_{n\to\infty} d(x_n, x_{n+1})=0.$$

Now from (F3), there exists $0 <k< 1$ such that,

$$\lim_{n\to\infty}[d(x_n, x_{n+1})]^k Fd(x_n, x_{n+1}) = 0.$$

By we have,

$$\lim_{n\to\infty}[d(x_n, x_{n+1})]^k[F\, d(x_n, x_{n+1}) - F(d(x_0, x_1))]$$

$$\leq \lim_{n\to\infty}[d(x_n, x_{n+1})]^k[(F(d(x_0, x_1)) - n\tau) - F(d(x_0, x_1)]$$

$$\leq -n\tau\, [d(x_n, x_{n+1})]^k$$

$$\leq 0.$$

By taking limit as $n \to\infty$ in above equation, we have

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^k = 0.$$

Hence $\lim_{n\to\infty} n^{1/k}\, d(x_n, x_{n+1})=0,$

now $\lim_{n\to\infty} n^{1/k}\, d(x_n, x_{n+1})=0.$

Ensures that the series $\sum_{n=1}^{\infty} d(x_n, x_{n+1})=0$. This implies that $\{x_n\}$ is a Cauchy sequence. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^* \in X$ such that, $x_n \to x^*$ as $n \to\infty$.

Since T is continuous, we have $Tx_n \to Tx^*$. By considering that the sequence $(d(x_{n+1}, Tx_n))$ is a constant sequence with value d(A, B),we deduce $d(x^*, Tx^*) = d(A, B)$. This means that x^* is a best proximity point of T. For uniqueness, suppose that x_1 and x_2 are two best proximity points of T with $x_1 \neq x_2$. This means that

$d(x_1, Tx_1) = d(A, B)$

$d(x_2, Tx_2) = d(A, B)$

Again, T is GF-proximal contraction, we have

$G\, (d(x_1, x_1), d(x_2, x_2), d(x_1, x_2), d(x_2, x_1)) + F\, d(x_1, x_2) \leq F\, d(x_1, x_2)$, implies

$\tau + Fd(x_1,x_2) \leq Fd(x_1,x_2)$,

which is contradiction. Hence $x_1 = x_2$.

Therefore T has a unique best proximity point.

Corollary 2.3: Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T: A \to B$ is a mapping for which there exists a function F belongs to family F and a contant $T > 0$ such that for each $x, y \in A$ and $u \in Tx$, $v \in Ty$ with

$d(u,Tx)=d(A,B)=d(v,Ty)$, we have

$\tau + F(d(u,v)) \leq F(d(x,y))$.

Further assume that the following condition hold, for each $x \in A_0$, we have $Tx \in B_0$. Then T has a best proximity point.

Example 2.4: Let X = R×R be endowed with a metric

$d((x_1, x_2),(y_1, y_2))=|x_1x_2|+|y_1y_2|$ for each $x,y \in X$.

Take $A=\{(0, x): -1 \le x \le 1\}$ and $B=\{(2,x): -1 \le x \le 1\}$.

Define $T:A \to B$, $T(0,x)= \begin{cases} \left(2, \frac{x+1}{2}\right), & \text{if } x \ge 0 \\ \{(2, x), (2, x)\}, & \text{otherwise} \end{cases}$

Take F = ln x for each $x \in (0, \infty)$ and $\tau = 1/8$.

When we take x = (0, 1/2), y = (0, 3/4) implies T(x) = (1, 3/4) and T(y) = (1, 7/8). We have d(u, Tx) = 2 = d(A, B) = d(v, Ty) for u = (0, 3/4) and v = (0, 7/8).

Now, L.H.S. of GF proximal contraction implies

$\tau + F((0, 3/4),(0, 7/8)) = 1/8\ 3 \ln 2$

and R.H.S. implies $F((0, 1/2),(0, 3/4))=F|(1/2) - 3/4| = F(1/4) = 2 \ln 2$

from above two equalities we obtain, T is GF proximal contraction.

Now, we introduce GF generalized proximal contraction and show that the existence of best proximity point for this contraction is unique.

Definition 2.5. Let (X, d) be a complete metric space and T be a nonself mapping from A to B where A and B are closed subset of complete metric space (X, d). T is said to be an GF- generalized proximal contraction if for $x, y \in X$ with $d(Tx, Ty) > 0$ we have,

$$G(d(x, u), d(y, v), d(x, v), d(y, u))+Fd(u, v) \le F(\alpha d(x, y) + \alpha d(x, u) + \gamma d(y, v) + \delta d(x, v) + Ld(y, u))$$

where $G \in \Delta G$ and $F \in \Delta F$, $\alpha, \beta, \gamma, \delta, L \ge 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$.

Theorem 2.6. Let(X, d) be complete metric space and A, B be two non empty closed subset of X. A_0, B_0 be nonempty subset of A and B respectively. Let $T: A \to B$ be a mapping satisfying following assertions:

I. T is continuous;

II. T has P-property;

III. T is an GF-generalized proximal contraction;

IV. $T(A_0) \subset B_0$

Then T has unique element $x \in A$ such that d(x, Tx) = d(A, B) and if $\{x_n\}$ is a sequence in A_0 satisfying $d(x_{n+1},Tx_n)=d(A,B)$ for all $n \ge 0$, then $\lim_{n\to\infty} x_n=x$.

Proof: Since A_0 is nonempty, we take $x_0 \in A$. As $Tx_0 \in T(A_0) \subset B_0$, we can find $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B).$$

Similarly, since $Tx_1 \in T(A_0) \subset B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Repeating this process, we can get a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for any } n \in N.$$

Since (A,B) has the P-property, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \text{ for any } n \in N.$$

As T is GF-proximal contraction,for any $n \in N$, we have that

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) + Fd(x_n, x_{n+1}) \leq F(ád$$
$$(x_{n-1}, x_n) + \beta d\,(x_{n-1}, x_n) + \gamma d\,(x_n, x_{n+1}) + \delta d\,(x_{n-1}, x_{n+1}) + Ld\,(x_n, x_n))$$

which implies

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) + F(d(x_n, x_{n+1})) \leq F(\alpha d\,(x_{n-1},x_n) + \beta d\,(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) + \delta d\,(x_{n-1}, x_{n+1}) + 0)$$

Now since, $d(x_{n-1}, x_n)\cdot d(x_n, x_{n+1})\cdot d(x_{n-1}, x_{n+1})\cdot 0 = 0$,

so from (G) there exists $T > 0$ such that,

$$Gd(x_{n-1},x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0 = \tau .$$

$$Fd(x_n, x_{n+1}) \leq F(\alpha d\,(x_{n-1}, x_n) + \beta d\,(x_{n-1}, x_n) + \gamma d\,(x_n, x_{n+1}) + \delta(d\,(x_{n-1}, x_n)$$
$$+ d\,(x_n,x_{n+1}))) - \tau$$
$$\leq F\,((\alpha + \beta + \delta)\,d\,(x_{n-1}, x_n) + (\gamma + \delta)\,d\,(x_n, x_{n+1})) - \tau$$

Since F is strictly increasing, we deduce that

$$d\,(x_n, x_{n+1}) < (\alpha + \beta + \delta)\,d\,(x_{n-1}, x_n) + (\gamma + \delta)\,d\,(x_n, x_{n+1}) .$$

This implies $(1- \gamma - \delta)\,d\,(x_n, x_{n+1}) < (\alpha + \beta + \delta)\,d\,(x_{n-1}, x_n)$ for all $n \in N$.

From $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma\neq 1$, we deduce that $1 - \gamma - \delta > 0$ and so

$d(x_n, x_{n+1}) < (\alpha + \beta + \delta) /(1 - \gamma - \delta)\,d\,(x_{n-1}, x_n) = d\,(x_{n-1}, x_n)$ for all $n \in N$. Consequently

$$F\,d(x_n, x_{n+1}) \leq F\,d(x_{n-1}, x_n) - \tau .$$

Continuing this process, we get

$Fd(x_n, x_{n+1}) \leq F\, d(x_{n-1}, x_n) - \tau.$

$\leq F\, d(x_{n-2}, x_{n-1}) - 2\tau \leq \ldots \leq F(d(x_0, x_1)) - n\tau.$

By taking limit as $n \to \infty$ in we have, $\lim_{n\to\infty} Fd(x_n, x_{n+1}) = -\infty$, and since, $F \in$ □ F we obtain,

$\lim_{n\to\infty} d(x_n, x_{n+1})=0.$

Now from (F3), there exists $0 <k< 1$ such that,

$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k F\, d(x_n, x_{n+1}) = 0.$

By we have,

$\lim_{n\to\infty} [d(x_n, x_{n+1})]^k[F\, d(x_n, x_{n+1}) - F(d(x_0, x_1))] \leq - n\tau\, [d(x_n, x_{n+1})]^k \leq 0.$

By taking limit as $n \to \infty$ in above equation, we have

$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^k = 0.$

Hence $\lim_{n\to\infty} n^{1/k} d(x_n, x_{n+1})=0$, now $\lim_{n\to\infty} n^{1/k} d(x_n, x_{n+1})=0.$

Ensures that the series $\Sigma_{n=1}^{\infty} d(x_n,x_{n+1})=0$. This implies that $\{x_n\}$ is a Cauchy sequence. Thus we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures that there exist $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$.

Since T is continuous ,we have $Tx_n \to Tx^*$. By considering that the sequence $(d(x_{n+1}, Tx_n))$ is a constant sequence with value d(A, B),we deduce $d(x^*, Tx^*) = d(A, B)$. This means that x^* is a best proximity point of T. For uniqueness, suppose that x_1 and x_2 are two best proximity points of T with $x_1=x_2$. This means that

$d(x_1, Tx_1) = d(A, B)$

$d(x_2, Tx_2) = d(A, B)$

Again, T is GF-generalized proximal contraction, we have

$$G(d(x_1,x_1),d(x_2,x_2),d(x_1,x_2),d(x_2,x_1)) + Fd(x_1,x_2) \leq F(\alpha d(x_1,x_2) + \beta d(x_1,x_1)+ \gamma d(x_2,x_2)+ \delta d(x_1,x_2)+Ld(x_2,x_1))$$

$$=F(\alpha+\delta+L)d(x_1,x_2).$$

Which is contradiction, if $\alpha+\delta+L \leq 1$ ad hence $x_1=x_2$.

Corollary 2.7. Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T: A \to B$ is a mapping for which there exists a function F belongs to family F and a constant $T > 0$ such that for each $x, y \in A$ and $u \in Tx$, $v \in Ty$ with

$d(u,Tx)=d(A,B)=d(v,Ty)$, we have

$\tau + F(d(u,v))\leq F(\alpha d(x,y) + \beta d(x,u) + \delta d(y,v) + \delta d(x,v)+Ld(y,u)).$

Where $F \in \Delta F$, α, β, γ, δ, $L \geq 0$; $\alpha+\beta+\gamma+2\delta=1$, $\gamma\neq 1$.

If for each $x \in A_0$, we have $Tx \in B_0$, then T has a best proximity point.

Example 2.8: when we take particular values of α, β, γ, δ and L that is $\beta=1$, γ, β, δ, L=0 in example 2.4 then we obtain GF generalized proximal contraction.

3. VARIATIONAL INEQUALITY PROBLEM

Now we consider the subsection, let H denote a real Hilbert space, with inner product and induced norm represented by <,> and ||.|| respectively. Let K be a non empty, closed and convex subset of H and $A:H\rightarrow H$ be a montone operator. We consider the following monotone variational inequality problem:

Find $u \in K$ such that <Au, v – >e"0 for all $v \in K$.
(3.1)

In early sixties variational inequality theory was introduced by Stamphaccia[16] and Fichera[17]. To solve the problem (3.1) we use the metric projection, say P_K: $H\rightarrow K$, which is a important tool for solving a variational inequality problem. Referring to Detuch[13], we contrive that for each $u\in H$, there exists a unique nearest point $P_K u\in K$ satisfying

$$\|u-P_K u\| \leq \|u-v\|, \text{ for all } v \in K.$$

Before proceding to check the link between variational inequality problem and special fixed point problem, we need the following lemma's.

Lemma 3.1 Let $z \in H$. Then $u \in K$ satisfies the inequality $\langle u-z, y-u\rangle \geq 0$, for all $y \in K$ if and only if $u = P_K z$.

Lemma 3.2 Let $A : H\rightarrow H$ be monotone. Then $u\in K$ is a solution of $(Au, v-u) \geq 0$, for all $v \in K$, if and only if $u = P_K(u-\lambda Au)$, with $\lambda > 0$.

On the basis of these lemma's, we prove some general convergence results on the solution of (3.1).

Theorem 3.3 Let K be a nonempty, closed, and convex subset of a real Hilbert space H and I_K be identity operator on K. Assume that the monotone operator $A:H\rightarrow H$ satisfies

(a) $P_K(I_K-\lambda A):K \rightarrow K$ is a GF proximal contraction with $\lambda > 0$.

(b) $P_K(I_K-\lambda A):K \rightarrow K$ is a GF generalized proximal contraction with $\lambda > 0$.

Then there exists a unique point $u \in K$ such that <Au, v-u> ³ 0 for all $v\in K$. Moreover, for each $u_0\in K$, there exist a sequence $\{u_n\}\in K$ such that $u_{n+1}=P_K(u_n-Au_n)$for every $n \in N \cup \{0\}$and $u_n\rightarrow u_0.$

Proof: Define an operator $T : K \to K$ by $Tx = P_K(x-\lambda Ax)$ for all $x \in K$. By Lemma 3.1, $u \in K$ is a solution of $<Au, v-u>$ e" 0 for all $v \in K$ if and only if $u = Tu$. As the operator T satisfies all the hypotheses of theorem 2.2 by taking $A = B = K$ and $G = \tau$. We conclude that theorem 3.3 hold true as an instant consequence of theorem 2.2. Similarly for the part(b), we obtain a unique fixed point by considering $G=\tau$, $\alpha=1,\beta=\gamma=\delta=L=0$.

Example 3.4 In the example 2.4 if we take $A = B = \{(0, x): -1\leq x \leq 1\}$ then we obtain a fixed point (0,1).

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