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# SOME PPF DEPENDENT FIXED POINT RESULTS FOR PREŠIĆ-HARDY-ROGERS CONTRACTIONS

**SAVITA RATHEE[1], NEELAM KUMARI[2] and MONIKA SWAMI[3]**

[1,2,3]Department of Mathematics
Maharshi Dayanand University
Rohtak (Haryana)-124001, India
E-mail: savitarathee.math@mdurohtak.ac.in
neelamjakhar45@gmail.com
monikaswani06@gmail.com

## Abstract

In this article, we develop some PPF dependent fixed point results for nonself mapping in Metric spaces for Prešić-Hardy-Rogers contraction, which is generalization of Prešić type contraction, where the domain space abstract is different from range space $E$. We also include some examples related to our results.

## 1. Introduction

Fixed point theory has several applications in various fields of research. It is a combination of analysis, topology and geometry. There has been a lot of research in this field since the establishment of the Banach contraction principle and some well-known fixed point theorems have emerged as an extension of this principle. It has been extended and generalized in many ways (see [1], [2], [5], [10], [11], [14], [19], [22], [23]). Several authors have dealt with the fixed point theory for different type of contractions in various spaces ([4], [6], [12], [13], [18]). After that, Prešić ([16], [17]) extended Banach contraction principle for mappings defined on product spaces and proved some fixed point results for the same.

Bernfeld et al. [3] developed an idea of a fixed point for mappings with

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distinct domains and ranges, known as the past-present-future (PPF) dependent fixed point or the fixed point with PPF dependence. They also introduced the concept of Banach type contraction for non-self mappings and demonstrated the existence of PPF dependent fixed point results in the Razumikhin class. These studies are valuable for establishing the solutions of nonlinear functional differential and integral equations that may depend upon past history, present data and future considerations. Many researchers have demonstrated several PPF dependent fixed point results (see [7], [8], [9], [20]).

Inspired by the work of Bernfeld et al. [3] and Shukla et al. [21], we develop some PPF dependent fixed point results for a nonself mapping in metric spaces for Prešić-Hardy-Rogers contraction which is generalization of Prešić type contraction.

Throughout this paper, $(E, d)$ is a complete metric space with the norm $\|\cdot\|_E$, $I$ is a closed interval $[a, b]$ in $\mathbb{R}$ and $E_0 = C(I, E)$ is the set of all continuous $E$-valued functions on $I$ with the corresponding metric

$$d_0(\psi, \xi) = \max_{c \in I} d[\psi(c), \xi(c)]. \tag{1.1}$$

And $\Omega_{\phi^*} = \{\psi \in E_0 : d_0(\psi, \phi^*) = d(\psi(c), \phi^*(c))\}$ is a class of functions in $E_0$. This class $\Omega_{\phi^*}$ is said to be algebraically closed with respect to difference if $\psi - \xi \in \Omega_{\phi^*}$ and topologically closed if it is closed with respect to topology on $E_0$ induced by $d_0$.

**Definition 2.1**[3]. "A function $\psi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of a nonself mapping $S$ if $S\psi = \psi(c)$ for some $c \in I$."

**Definition 2.2.** "Let $(E, d)$ be a metric space, $l$ be a positive integer and $S : E_0^l \to E$ be a nonself mapping then

1. [16] $S$ is said to be a Prešić contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1})) \leq \sum_{j=l}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c))$$

where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are non negative constants such that $\alpha_1 + \alpha_2 + \ldots + \alpha_l < 1$.

2. [15] $S$ is said to be Prešić-Kannan contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1})) \leq \beta \sum_{j=l}^{l+1} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j))$$

where $0 \leq \beta l(l+1) < 1$.

3. [18] $S$ is said to be Prešić-Reich contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1})) \leq \sum_{j=l}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c))$$

$$+ \beta_j \sum_{j=l}^{l+1} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j))$$

where $\alpha_j, \beta_j$ are non negative constants such that $\sum_{j=1}^{l} \alpha_j + l \sum_{j=1}^{l+1} \beta_j < 1$.

4. [4] $S$ is said to be a Prešić-Chatterjea contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1}))$$

$$\leq \gamma \sum_{j=1, j \neq k}^{l+1} \sum_{k=1}^{l+1} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j))$$

where $0 \leq \gamma l^2(l+1) < 1$.

5. [6] $S$ is said to be Generalized-Prešić contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1}))$$

$$\leq \sum_{j=1}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c)) + \beta_j \sum_{j=1}^{l+1} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j))$$

$$+ \beta \sum_{j=1, j \neq k}^{l+1} \sum_{k=1}^{l+1} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j)),$$

where $\alpha_j, \beta_j, \beta$ are non negative constants such that

$$\sum_{j=1}^{l} \alpha_j + l \sum_{j=1}^{l+1} \sum_{k=1}^{l+1} \beta_j + \beta l^2 (l+1) < 1.$$

6. [11] $S$ is said to be Prešić-Hardy-Rogers contraction if it satisfies

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3, \ldots, \psi_{l+1}))$$

$$\leq \sum_{j=1}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c)) + \sum_{j=1}^{l+1} \sum_{k=1}^{l+1} \beta_{j,k} d(\psi_j(c), S(\psi_j, \psi_j, \ldots, \psi_j))$$

where $\alpha_j, \beta_{j,k}$ are non negative constants such that

$$\sum_{j=1}^{l} \alpha_j + l \sum_{j=1}^{l+1} \sum_{k=1}^{l+1} \beta_{j,k} < 1,$$

for all $\psi_1, \psi_2, \ldots, \psi_k, \psi_{k+1} \in E_0$."

## 2. The Main Results

**Theorem 3.1.** *Let $(E, d)$ be a complete metric space and $I = [a, b]$ be any closed interval in $\mathbb{R}$. Suppose $E_0 = C(I, E)$ denotes the set of all continuous function on I to $E$, $S : E_0^l \to E$ is a Prešić contraction and $\Omega_{\phi^*}$ is a class of functions in $E_0$, which is topologically and algebraically closed with respect to difference. Then, S has a unique PPF dependent fixed point in $\Omega_{\phi^*}$.*

**Proof.** Let $\psi_0 \in \Omega_{\psi^*} \subseteq E_0$. Clearly $S(\psi_0, \ldots, \psi_0) \in E$. Let us suppose $S(\psi_0, \ldots, \psi_0) = x_1$. Define $\psi_1 : I \to E$ as $\psi_1(z) = x_1$ for some $z \in I$. Then $\psi_1 \in E_0$. We choose $\psi_1 \in \Omega_{\phi^*}$ s.t. $S(\psi_0, \ldots, \psi_0) = \psi_1(c) = x_1$. Let $S(\psi_1, \ldots, \psi_1) = x_2$. Consider $\psi_2 : I \to E$ as $\psi_2(z) = x_2$ for some $z \in I$. Then $\psi_2 \in E_0$. Choose $\psi_2 \in \Omega_{\phi^*}$ s.t. $S(\psi_1, \ldots, \psi_1) = \psi_2(c) = x_2$. Let $S(\psi_2, \ldots, \psi_2) = x_3$. We define $\psi_3 : I \to E$ as $\psi_3(z) = x_3$ for some $z \in I$.

Then, $\psi_3 \in E_0$. Hence we take $\psi_3 \in \Omega_{\phi^*}$ s.t. $S(\psi_2, \ldots, \psi_2) = x_3 = \psi_3(c)$. Continuing this process, we define a sequence $\{\psi_n\}$ s.t.

$$S(\psi_n, \ldots, \psi_n) = x_{n+1} = \psi_{n+1}(c) \text{ for } n \in \{0, 1, 2, \ldots\}.$$

If $\psi_{n+1} = \psi_n$ for some $n \in \{0, 1, 2, \ldots\}$, then

$$S(\psi_n, \ldots, \psi_n) = \psi_{n+1}(c) = \psi_n(c).$$

Thus $\psi_n$ is a PPF dependent fixed point of $S$ in $\Omega_{\phi^*}$. So we assume

$$\psi_{n+1} \neq \psi_n \ \forall n \in \{0, 1, 2, \ldots\}.$$

For our convenience, let

$$d_j = d(\psi_j(c), \psi_{j+1}(c)) \text{ and } D_{j,k} = d(\psi_j(c), S(\psi_k, \ldots, \psi_k)) \ \forall j, k \geq 1 \tag{2.1}$$

We now prove that $\{\psi_n\}$ is a Cauchy sequence. For $n \in \{0, 1, 2, \ldots\}$, consider

$$\begin{aligned} d_{n+1} = d(\psi_{n+1}(c), \psi_{n+2}(c)) \\ &= d(S(\psi_n, \ldots, \psi_n), S(\psi_{n+1}, \ldots, \psi_{n+1})) \\ &\leq d(S(\psi_n, \ldots, \psi_n), S(\psi_n, \ldots, \psi_n, \psi_{n+1})) \\ &+ d(S(\psi_n, \ldots, \psi_n, \psi_{n+1}), S(\psi_n, \ldots, \psi_n, \psi_{n+1}, \psi_{n+1})) \\ &+ \ldots + d(S(\psi_n, \psi_{n+1}, \ldots, \psi_{n+1}), S(\psi_{n+1}, \ldots, \psi_{n+1})) \end{aligned}$$

By Prešić contraction

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_n, \psi_n, \ldots, \psi_{l+1})) \leq \sum_{j=l}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c))$$

for all $\psi_1, \psi_2, \ldots, \psi_l, \psi_{l+1} \in E_0$, and $\alpha_j \geq 0$ such that $\sum_{j=l}^{l} \alpha_j < 1$.

So, $d_{n+1} \leq \alpha_l d_n + \alpha_{l-1} d_n + \ldots + \alpha_1 d_n$.

Thus,

$$d_{n+1} \leq \left[\sum_{j=1}^{l} \alpha_j\right] d_n.$$

Now, take $\alpha_1 + \alpha_2 + \ldots + \alpha_l = \mu$. So, $d_{n+1} \leq \mu d_n$.

Clearly $\mu < 1$.

So, we get

$$d_{n+1} \leq \mu^{n+1} d_0 \tag{2.2}$$

As $d_{n+1} = d(\psi_n(c), \psi_{n+1}(c)) = d(\psi_{n+1}(c), \psi_n(c))$.

Let if possible $\{\psi_n\}$ is not Cauchy, then $\exists$ an $\delta > 0$ and sequence of positive integers $p$ and $q$ with $p > q$ such that

$$d(\psi_p(c), \psi_q(c)) \geq \delta \text{ and } d(\psi_p(c), \psi_{q-1}(c)) \leq \delta.$$

Now,

$$\delta \leq d(\psi_p(c), \psi_q(c))$$

$$\leq d(\psi_p(c), \psi_{p+1}(c)) + d(\psi_{p+1}(c), \psi_{p+2}(c)) + \ldots + d(\psi_{q-1}(c), \psi_q(c))$$

$$= d_p + d_{p+1} + \ldots + d_{q-1}$$

$$\leq \mu^p d_0 + \mu^{p+1} d_0 + \ldots + \mu^{q-1} d_0$$

$$\leq \frac{\mu^p}{1-\mu} d_0.$$

Now, $0 \leq \mu < 1$. So, by appling $q \to \infty$, we get $\lim_{q\to\infty} d(\psi_p(c), \psi_q(c)) = 0$. Hence $\delta = 0$,

This is a contradiction.

So, $\psi_n$ is a Cauchy sequence in $\Omega_{\phi^*} \subseteq E_0$. We take $\lim_{n\to\infty} \psi_n = \psi^*$.

Since $E_0$ is a complete. So, we have $\psi_n$ is convergent. Thus, $\psi^* \in E_0$.

Now, $\psi^* \in \Omega_{\phi^*}$, because $\Omega_{\phi^*}$ is topologically closed.

We prove that $\psi^*$ is a PPF dependent fixed point of $S$. We consider

$$d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) \leq d(\psi^*(c), \psi_{n+1}(c)) + d(\psi_{n+1}(c), S(\psi^*, \ldots, \psi^*))$$

$$= d(\psi^*(c), \psi_{n+1}(c)) + d(S(\psi_n, \ldots, \psi_n), S(\psi^*, \ldots, \psi^*)).$$

By the same method as used in the calculation of $d_{n+1}$, we get

$$d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) \leq d(\psi^*(c), \psi_{n+1}(c)) + \mu d(\psi_n(c), \psi^*(c)).$$

By using $\lim_{n\to\infty} = \psi^*$, we have $d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) = 0$. So, $S(\psi^*, \ldots, \psi^*) = \psi^*(c)$.

Hence $\psi^*$ is a PPF dependent fixed point of $S$. For uniqueness, let $\xi^*$ be any other PPF dependent fixed point of $S$, that is, $S(\xi^*, \ldots, \xi^*) = \xi^*$. Again by the similar process as used in the calculation of $d_{n+1}$, we get $d(\psi^*, \xi^*) = 0$. Hence $\psi^* = \xi^*$.

Thus PPF dependent fixed point is unique.

**Theorem 3.2.** *Let $(E, d)$ be a complete metric space and $I = [a, b]$ be any closed interval in $\mathbb{R}$. Suppose $E_0 = C(I, E)$ denotes the set of all continuous function on $I$ to $E$, $S : E_0^l \to E$ is a Prešić-Hardy-Rogers contraction and $\Omega_{\phi^*}$ is a class of functions in $E_0$, which is topologically and algebraically closed with respect to difference. Then, $S$ has only one PPF dependent fixed point in $\Omega_{\phi^*}$.*

**Proof.** Let $\psi_0 \in \Omega_{\phi^*} \subseteq E_0$. Clearly $S(\psi_0, \ldots, \psi_0) \in E$. Let us suppose $S(\psi_0, \ldots, \psi_0) = x_1$.

Define $\psi_1 : I \to E$ as $\psi_1(z) = x_1$ for some $z \in I$, then $\psi_1 \in E_0$. We choose $\psi_1 \in \Omega_{\phi^*}$ s.t. $S(\psi_0, \ldots, \psi_0) = \psi_1(c) = x_1$. Let $S(\psi_1, \ldots, \psi_1) = x_2$. Now define $\psi_2 : I \to E$ as $\psi_2(z) = x_2$ for some $z \in I$, then $\psi_2 \in E_0$. We take $\psi_2 \in \Omega_{\phi^*}$ s.t. $S(\psi_1, \ldots, \psi_1) = \psi_2(c) = x_2$. Let $S(\psi_2, \ldots, \psi_2) = x_3$. Define $\psi_3 : I \to E$ as $\psi_3(z) = x_3$ for some $z \in I$. Then $\psi_3 \in E_0$. Hence choose $\psi_3 \in \Omega_{\phi^*}$ s.t. $S(\psi_2, \ldots, \psi_2) = x_3 = \psi_3(c)$. Continuing this process, we define a sequence $\{\psi_n\}$ s.t. $S(\psi_n, \ldots, \psi_n) = x_{n+1} = \psi_{n+1}(c)$ for $n \in \{0, 1, 2, \ldots\}$.

If $\psi_{n+1} = \psi_n$ for some $n \in \{0, 1, 2, \ldots\}$, then

$$S(\psi_n, \ldots, \psi_n) = \psi_{n+1}(c) = \psi_n(c).$$

Thus $\psi_n$ is a PPF dependent fixed point of $S$ in $\Omega_{\phi^*}$. So, we assume $\psi_{n+1} \neq \psi_n \ \forall\, n \in \{0, 1, 2, \ldots\}$.

For our convenience, let

$$d_j = d(\psi_j(c), \psi_{j+1}(c)) \text{ and } D_{j,k} = (\psi_j(c), S(\psi_k, \ldots, \psi_k)) \ \forall\, j, k \geq 1 \tag{2.3}$$

We now prove that $\psi_n$ is a Cauchy sequence. For $n \in \{0, 1, 2, \ldots\}$

$$\begin{aligned} d_{n+1} &= d(\psi_{n+1}(c), \psi_{n+2}(c)) \\ &= d(S(\psi_n, \ldots, \psi_n), S(\psi_{n+1}, \ldots, \psi_{n+1})) \\ &\leq d(S(\psi_n, \ldots, \psi_n), S(\psi_n, \ldots, \psi_n, \psi_{n+1})) \\ &\quad + d(S(\psi_n, \ldots, \psi_n, \psi_{n+1}), S(\psi_n, \ldots, \psi_n,. \psi_{n+1}, \psi_{n+1})) \\ &\quad + \ldots + d(S(\psi_n, \psi_{n+1}, \ldots, \psi_{n+1}), S(\psi_{n+1}, \ldots, \psi_{n+1})). \end{aligned}$$

By Prešić-Hardy-Rogers contraction

$$d(S(\psi_1, \psi_2, \ldots, \psi_l), S(\psi_2, \psi_3 \ldots, \psi_{l+1})) \leq \sum_{j=1}^{l} \alpha_j d(\psi_j(c), \psi_{j+1}(c))$$

$$+ \sum_{j=1}^{l+1} \sum_{k=1}^{l+1} \beta_{j,k} d(\psi_j(c), S(\psi_k, \psi_k, \ldots, \psi_k))$$

for all $\psi_1, \psi_2, \psi_l, \psi_{l+1} \in E_0$.

Where $\alpha_j, \beta_{j,k} \geq 0$ such that

$$\sum_{j=1}^{l} \alpha_j + l \sum_{j=1}^{l+1} \sum_{k=1}^{l+1} \beta_{j,k} < 1.$$

So,

$$d_{n+1} \leq \left\{ \alpha_l d_n + \left[ \sum_{k=1}^{l} \beta_{1,k} + \sum_{k=1}^{l} \beta_{2,k} + \ldots + \sum_{k=1}^{l} \beta_{l,k} \right] D_{n,n} + \left[ \sum_{j=1}^{l} \beta_{j,l+1} \right] D_{n,n+1} \right.$$

$$\left. + \left[ \sum_{k=1}^{l} \beta_{l+1,k} \right] D_{n+1,n} + \beta_{l+1,l+1} D_{n+1,n+1} \right\} + \left\{ \alpha_{l-1} d_n + \left[ \sum_{k=1}^{l-1} \beta_{1,k} \right. \right.$$

$$\left. + \sum_{k=1}^{l-1} \beta_{2,k} + \ldots + \sum_{k=1}^{l-1} \beta_{l-1,k} \right] D_{n,n} + \left[ \sum_{j=1}^{l-1} \beta_{j,l} + \sum_{j=1}^{l-1} \beta_{j,l+1} \right] D_{n,n+1}$$

$$\left. + \left[ \sum_{k=1}^{l-1} \beta_{l,k} + \sum_{k=1}^{l-1} \beta_{l+1,k} \right] D_{n+1,n} + \left[ \sum_{k=1}^{l-1} \beta_{l,k} + \sum_{k=1}^{l-1} \beta_{l+1,k} \right] D_{n+1,n+1} \right\}$$

$$+ \left\{ \alpha_1 d_n + \beta_{1,1} D_{n,n} + \left[ \sum_{k=2}^{l-1} \beta_{1,k} \right] D_{n,n+1} + \left[ \sum_{j=2}^{l+1} \beta_{j,1} \right] D_{n+1,n} \right.$$

$$\left. + \left[ \sum_{k=2}^{l+1} \beta_{2,k} + \sum_{k=2}^{l+1} \beta_{3,k} + \ldots + \sum_{k=2}^{l+1} \beta_{l+1,k} \right] D_{n+1} D_{n+1} \right\}$$

that is

$$d_{n+1} \leq \left[ \sum_{j=1}^{l} \alpha_j \right] d_n + \left\{ \left[ \sum_{j=1}^{l} \sum_{k=1}^{l} \beta_{j,k} \right] D_{n,n} + \left[ \sum_{j=1}^{l} \beta_{j,l+1} \right] D_{n,n+1} \right.$$

$$\left. + \left[ \sum_{k=1}^{l} \beta_{l+1,k} \right] D_{n+1,n} + \beta_{l+1,l+1} D_{n+1,n+1} \right\}$$

$$+\left\{\left[\sum_{j=1}^{l-1}\sum_{k=1}^{l-1}\beta_{j,k}\right]D_{n,n}+\left[\sum_{j=1}^{l-1}\sum_{k=l}^{l-1}\beta_{j,k}\right]D_{n,n+1}\right.$$

$$\left.+\left[\sum_{j=1}^{l+1}\sum_{k=l}^{l-1}\beta_{j,k}\right]D_{n+1,n}+\left[\sum_{j=l}^{l+1}\sum_{k=l}^{l+1}\beta_{j,k}\right]D_{n+1,n+1}\right\}$$

$$+\ldots+\left\{\beta_{1,1}D_{n,n}+\left[\sum_{k=2}^{l+1}\beta_{1,k}\right]D_{n,n+1}+\left[\sum_{j=2}^{l+1}\beta_{j,1}\right]D_{n+1,n}\right.$$

$$\left.+\left[\sum_{j=2}^{l+1}\sum_{k=2}^{l+1}\beta_{j,k}\right]D_{n+1,n+1}\right\}$$

that is

$$d_{n+1}\le\left[\sum_{j=1}^{l}\alpha_j\right]d_n$$

$$+\left[\sum_{j=1}^{l}\sum_{k=1}^{l}\beta_{j,k}+\sum_{j=1}^{l-1}\sum_{k=1}^{l-1}\beta_{j,k}+\ldots+\sum_{j=1}^{2}\sum_{k=1}^{2}\beta_{j,k}+\beta_{1,1}\right]D_{n,n}$$

$$+\left[\sum_{k=1}^{l}\beta_{j,l+1}+\sum_{j=1}^{l-1}\sum_{k=1}^{l-1}\beta_{j,k}+\ldots+\sum_{j=1}^{2}\sum_{k=2}^{l+1}\beta_{j,k}+\sum_{k=2}^{l+1}\beta_{1,k}\right]D_{n,n+1}$$

$$+\left[\sum_{k=1}^{l}\beta_{l+1,k}+\sum_{j=1}^{l+1}\sum_{k=1}^{l-1}\beta_{j,k}+\ldots+\sum_{j=3}^{l+1}\sum_{k=1}^{2}\beta_{j,k}+\sum_{j=2}^{l+1}\beta_{j,1}\right]D_{n+1,n}$$

$$+\left[\sum_{j=2}^{l+1}\sum_{k=2}^{l+1}\beta_{j,k}+\sum_{j=3}^{l+1}\sum_{k=3}^{l+1}\beta_{j,k}+\ldots+\sum_{j=l}^{l+1}\sum_{k=l}^{l+1}\beta_{j,k}+\beta_{l+1,l+1}\right]D_{n+1,n+1}$$

$$=C_1d_n+C_2D_{n,n}+C_3D_{n,n+1}+C_4D_{n+1,n}+C_5D_{n+1,n+1}$$

where $C_1, C_2, C_3, C_4, C_5$ are the coefficients of $d_n, D_{n,n}, D_{n,n+1}, D_{n+1,n}$ and $D_{n+1,n+1}$ respectively.

Now,

$$D_{n,n} = d(\psi_n(c), S(\psi_n, \ldots, \psi_n)) = d(\psi_n(c), \psi_{n+1}(c)) = d_n;$$

$$D_{n,n+1} = d(\psi_n(c), S(\psi_{n+1}, \ldots, \psi_{n+1})) = d(\psi_n(c), \psi_{n+2}(c));$$

$$D_{n+1,n} = d(\psi_{n+1}(c), S(\psi_n, \ldots, \psi_n)) = d(\psi_{n+1}(c), \psi_{n+1}(c)) = 0;$$

$$D_{n+1,n+1} = d(\psi_{n+1}(c), S(\psi_{n+1}, \ldots, \psi_{n+1})) = d(\psi_{n+1}(c), \psi_{n+2}(c)) = d_{n+1}.$$

Thus,

$$d_{n+1} \le C_1 d_n + C_2 d_n + C_3 d(\psi_n(c), \psi_{n+2}(c)) + C_5 d_{n+1}$$

$$\le C_1 d_n + C_2 d_n + C_3 d(\psi_n(c), \psi_{n+1}(c)) + C_3 d(\psi_{n+1}(c), \psi_{n+2}(c)) + C_5 d_{n+1}$$

$$\le (C_1 + C_2 + C_3) d_n + (C_3 + C_5) d_{n+1}$$

that is

$$(1 - C_3 - C_5) d_{n+1} \le (C_1 + C_2 + C_3) d_n. \tag{2.4}$$

As $d_{n+1} = d(\psi_n(c), \psi_{n+1}(c)) = d(\psi_{n+1}(c), \psi_n(c))$

If we interchange the role of $\psi_n$ and $\psi_{n+1}$ then by above process, we have

$$(1 - C_4 - C_2) d_{n+1} \le (C_1 + C_5 + C_4) d_n \tag{2.5}$$

By (2.4) and (2.5)

$$(2 - C_2 - C_3 - C_4 - C_5) d_{n+1} \le (2C_1 + C_2 + C_3 + C_4 + C_5) d_n$$

$$d_{n+1} \le \frac{(2C_1 + C_2 + C_3 + C_4 + C_5)}{(2 - C_2 - C_3 - C_4 - C_5)} d_n$$

If we take $\mu = \frac{(2C_1 + C_2 + C_3 + C_4 + C_5)}{(2 - C_2 - C_3 - C_4 - C_5)}$, then

$$d_{n+1} \le \mu d_n \tag{2.6}$$

By using

$$\sum_{j=1}^{l}\alpha_j + l\sum_{j=1}^{l+1}\sum_{k=1}^{l+1}\beta_{j,k} < 1,$$

we have

$C_1 + C_2 + C_3 + C_4 + C_5$

$$= \sum_{j=1}^{l}\alpha_j + \sum_{j=1}^{l}\sum_{k=1}^{l}\beta_{j,k} + \sum_{j=1}^{l-1}\sum_{k=1}^{l-1}\beta_{j,k} + \ldots + \sum_{j=1}^{2}\sum_{k=1}^{2}\beta_{j,k} + \beta_{1,1}$$

$$+ \sum_{j=1}^{l}\beta_{j,l+1} + \sum_{j=1}^{l-1}\sum_{k=1}^{l+1}\beta_{j,k} + \ldots + \sum_{j=1}^{2}\sum_{k=3}^{l+1}\beta_{j,k} + \sum_{k=2}^{l+1}\beta_{1,k}$$

$$+ \sum_{j=1}^{l}\beta_{l+1,k} + \sum_{j=l}^{l+1}\sum_{k=l}^{l-1}\beta_{j,k} + \ldots + \sum_{j=3}^{l+1}\sum_{k=1}^{2}\beta_{j,k} + \sum_{j=2}^{l+1}\beta_{j,1}$$

$$+ \sum_{j=2}^{l+1}\sum_{k=2}^{l+1}\beta_{j,k} + \sum_{j=3}^{l+1}\sum_{k=3}^{l+1}\beta_{j,k} + \ldots + \sum_{j=l}^{l+1}\sum_{k=l}^{l+1}\beta_{j,k} + \beta_{l+1,l+1}$$

$$= \sum_{j=1}^{l}\alpha_j + l\sum_{j=1}^{l+1}\sum_{k=1}^{l+1}\beta_{j,k} < 1.$$

Thus $0 \le \mu < 1$. By using (2.6),

$$d_{n+1} \le \mu^{n+1} d_0 \ \forall\ n \ge 0.$$

Let if possible $\{\psi_n\}$ is not Cauchy, then $\exists$ an $\delta > 0$ and sequence of positive integers $p$ and $q$ with $p > q$ such that

$$d(\psi_p(c), \psi_q(c)) \ge \delta \text{ and } d(\psi_p(c), \psi_{q-1}(c)) \le \delta.$$

Now

$\delta \le d(\psi_p(c), \psi_q(c))$

$d(\psi_p(c), \psi_{p+1}(c)) + d(\psi_{p+1}(c), \psi_{p+2}(c)) + \ldots + d(\psi_{q-1}(c), \psi_q(c))$

$$= d_p + d_{p+1} + \ldots + d_{q-1}$$

$$\leq \mu^p d_0 + \mu^{p+1} d_0 + \ldots + \mu^{q-1} d_0$$

$$\leq \frac{\mu^p}{1-\mu} d_0.$$

Now $0 \leq \mu < 1$. So, by appling $q \to \infty$, we get $\lim_{q\to\infty} d(\psi_p(c), \psi_p(c)) = 0$. Hence $\delta = 0$,

Here, we have a contradiction.

So $\psi_n$ is a Cauchy sequence in $\Omega_{\phi^*} \subseteq E_0$. We take $\lim_{n\to\infty} \psi_n = \psi^*$.

Since $E_0$ is a complete. So, we have $\psi_n$ is convergent. So, $\psi^* \in E_0$.

Now $\psi^* \in \Omega_{\phi^*}$, because $\Omega_{\phi^*}$ is topologically closed.

Now we demonstrate that $\psi^*$ is a PPF dependent fixed point of $S$. We consider

$$d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) \leq d(\psi^*(c), \psi_{n+1}(c)) + d(\psi_{n+1}(c), S(\psi^*, \ldots \psi^*))$$

$$= d(\psi^*(c), \psi_{n+1}(c)) + d(S(\psi_n, \ldots, \psi_n), S(\psi^*, \ldots, \psi^*)).$$

By the same method as used in the calculation of $d_{n+1}$, we get

$$d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) \leq d(\psi^*(c), \psi_{n+1}(c)) + C_1 d(\psi_n(c), \psi^*(c))$$

$$+ C_2 d(\psi_n(c), S(\psi_n, \ldots, \psi_n) + C_3 d(\psi_n(c), S(\psi^*, \ldots, \psi^*))$$

$$+ C_4 d(\psi^*(c), S(\psi_n, \ldots, \psi_n)) + C_5 d(\psi^*(c), S(\psi^*, \ldots, \psi^*))$$

$$\leq d(\psi^*(c), \psi_{n+1}(c)) + C_1 d(\psi_n(c), \psi^*(c)) + C_2 d(\psi_n(c), \psi_{n+1}(c))$$

$$+ C_3 d(\psi_n(c), \psi^*(c) + C_3 d(\psi^*(c) S(\psi^*, \ldots, \psi^*)) + C_4 d(\psi^*(c), \psi_{n+1}(c))$$

$$+ C_5 d(\psi^*(c), S(\psi^*, \ldots, \psi^*))$$

that implies

$$d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) \leq \frac{C_1 + C_2 + C_3}{1 - C_3 - C_5} d(\psi_n(c), \psi^*(c))$$

$$+ \frac{C_1 + C_2 + C_4}{1 - C_3 - C_5} d(\psi_{n+1}(c), \psi^*(c)).$$

By using $\lim_{n \to \infty} = \psi^*$, we have $d(\psi^*(c), S(\psi^*, \ldots, \psi^*)) = 0$. So, $S(\psi^*, \ldots, \psi^*) = \psi^*(c)$.

Hence $\psi^*$ is a PPF dependent fixed point of $S$.

For uniqueness, consider $\xi^*$ is any other PPF dependent fixed point of $S$, that is, $S(\xi^*, \ldots, \xi^*) = \xi^*$. Again by the same process as used in the calculation of $d_{n+1}$,

$$d(\psi^*, \xi^*) \leq C_1 d(\psi^*, \xi^*) + C_2 d(\psi^*, S(\psi^*, \ldots, \psi^*)) + C_3 d(\psi^*, S(\xi^*, \ldots, \xi^*))$$

$$+ C_4 d(\xi^*, S(\psi^*, \ldots, \psi^*)) + C_5 d(\xi^*, S(\xi^*, \ldots, \xi^*))$$

$$= (C_1 + C_3 + C_4) d(\psi^*, \xi^*).$$

As, $C_1 + C_2 + C_3 + C_4 + C_5 < 1$. So, $d(\psi^*, \xi^*) = 0$. Hence $\psi^* = \xi^*$.

Thus PPF dependent fixed point is unique.

**Example 1.** Let $E = \mathbb{R}$ and $E_0 = C(I, \mathbb{R})$ where $I = [0, 1]$. Fix a point $c = \frac{1}{3} \in [0, 1]$. Let us define $S : E_0 \times E_0 \to E$ by

$$S(\psi, \psi) = \frac{5}{9} \psi\left(\frac{1}{3}\right) + \frac{4}{81}, \phi \in E_0$$

$$\psi(x) = \begin{cases} x^2 & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \frac{1}{9} & \text{if } x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

Now

$$S(\psi, \psi) = \frac{5}{9}\psi\left(\frac{1}{3}\right) + \frac{4}{81} = \frac{5}{81} + \frac{4}{81} = \frac{9}{81} = \frac{1}{9} \text{ and } \psi\left(\frac{1}{3}\right) = \frac{1}{9}$$

Here $S(\psi, \psi) = 1/9 = \psi(1/3)$.

Thus, $\psi$ is a PPF dependent fixed point of $S$.

**Example 2.** Let $S : E_0 \times E_0 \to E$ be nonself mapping where $(E, d)$ is a complete metric space and $I = [0, 1] \in \mathbb{R}$. We define $S$ by

$$S(\phi, \psi) = \frac{\phi(c) + \psi(c)}{5}$$

and $\phi_1, \phi_2, \phi_3 : I \to E$ by

$$\phi_1(c) = 1, \forall x \in [0, 1]$$

$$\phi_2(c) = \begin{cases} x^2 & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{4} & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\phi_3(c) = \begin{cases} x^2 & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \frac{1}{9} & \text{if } x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

We show that $S$ is a Prešić-Hardy-Rogers contraction.

So, we have to prove that

$$\begin{aligned} d(S(\phi_1, \phi_2), S(\phi_2, \phi_3)) &\leq \alpha_1 d(\phi_1(c), \phi_2(c)) + \alpha_2 d(\phi_2(c), \phi_3(c) \\ &+ \beta_{1,1} d(\phi_1(c), S(\phi_1, \phi_1)) \\ &+ \beta_{1,2} d(\phi_1(c), S(\phi_2, \phi_2)) + \beta_{1,3} d(\phi_1(c), S(\phi_3, \phi_3)) + \beta_{2,1} d(\phi_2(c), S(\phi_1, \phi_1)) \\ &+ \beta_{2,2} d(\phi_2(c), S(\phi_2, \phi_2)) + \beta_{2,3} d(\phi_2(c), S(\phi_3, \phi_3)) + \beta_{3,1} d(\phi_3(c), S(\phi_1, \phi_1)) \\ &+ \beta_{3,2} d(\phi_3(c), S(\phi_2, \phi_2)) + \beta_{3,3} d(\phi_3(c), S(\phi_3, \phi_3)). \end{aligned} \tag{2.7}$$

Now

$$d(S(\phi_1, \phi_2), S(\phi_2, \phi_3)) = d\left(\frac{\phi_1(c) + \phi_2(c)}{5} + \frac{\phi_2(c) + \phi_3(c)}{5}\right)$$

$$= \frac{\phi_1(c) - \phi_3(c)}{5}$$

$$= \begin{cases} \dfrac{1 - x^2}{5} & \text{if } x \in \left[0, \dfrac{1}{3}\right] \\ \dfrac{8}{45} & \text{if } x \in \left[0, \dfrac{1}{3}\right] \end{cases}$$

and

$$\alpha_1 d(\phi_1(c), \phi_2(c)) = \begin{cases} \alpha_1(1 - x^2) & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ \alpha_1\left(\dfrac{3}{4}\right) & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

$$\alpha_2 d(\phi_2(c), \phi_3(c)) = \begin{cases} 0 & \text{if } x \in \left[0, \dfrac{1}{3}\right] \\ \alpha_2\left(x^2 - \dfrac{1}{9}\right) & \text{if } x \in \left[\dfrac{1}{3}, \dfrac{1}{2}\right] \\ \alpha_2\left(\dfrac{5}{36}\right) & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

$$\beta_{1,1} d(\phi_1(c), S(\phi_1, \phi_1)) = \beta_{1,1} d\left(1, \frac{2\phi_1(c)}{5}\right) = \beta_{1,1}\left(1 - \frac{2\phi_1(c)}{5}\right)$$

$$= \beta_{1,1}\left(1 - \frac{2}{5}\right) = \beta_{1,1}\left(\frac{3}{5}\right)$$

$$\beta_{1,2} d(\phi_1(c), S(\phi_2, \phi_2)) = \beta_{1,2} d\left(1, \frac{2\phi_2(c)}{5}\right) = \beta_{1,2}\left(1 - \frac{2\phi_2(c)}{5}\right)$$

$$= \begin{cases} \beta_{1,2}\left(1 - \dfrac{2x^2}{5}\right) & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ \beta_{1,2}\left(\dfrac{9}{10}\right) & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

$$\beta_{1,3} d(\phi_1(c), S(\phi_3, \phi_3)) = \beta_{1,3} d\left(1, \frac{2\phi_3(c)}{5}\right) = \beta_{1,3}\left(1 - \frac{2\phi_3(c)}{5}\right)$$

$$= \begin{cases} \beta_{1,3}\left(1 - \frac{2x^2}{5}\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \beta_{1,3}\left(\frac{43}{45}\right) & \text{if } x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

$$\beta_{2,1}d(\phi_2(c), S(\phi_1, \phi_1)) = \beta_{2,1}d\left(\phi_2(c), \frac{2\phi_1(c)}{5}\right)$$

$$= \beta_{2,1}\left(\phi_2(c), \frac{2}{5}\right) = \begin{cases} \beta_{2,1}\left|\left(x^2 - \frac{2}{5}\right)\right| & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \beta_{2,1}\left(\frac{3}{20}\right) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\beta_{2,2}d(\phi_2(c), S(\phi_2, \phi_2)) = \beta_{2,2}d\left(\phi_2(c), \frac{2\phi_2(c)}{5}\right)$$

$$= \beta_{2,2}\left(\frac{3}{5}\phi_2(c)\right) = \begin{cases} \beta_{2,2}\frac{3}{5}(x^2) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \beta_{2,2}\left(\frac{3}{20}\right) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\beta_{2,3}d(\phi_3(c), S(\phi_3, \phi_3)) = \beta_{2,3}d\left(\phi_3(c), \frac{2\phi_3(c)}{5}\right)$$

$$= \begin{cases} \beta_{2,3}\left(\frac{3}{2}x^2\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \beta_{2,3}\left(x^2 - \frac{2}{45}\right) & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right] \\ \beta_{2,3}\left(\frac{37}{180}\right) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\beta_{3,1}d(\phi_3(c), S(\phi_1, \phi_1)) = \beta_{3,1}d\left(\phi_3(c), \frac{2\phi_1(c)}{5}\right)$$

$$= \begin{cases} \beta_{3,1}\left|\left(x^2 - \frac{3}{5}\right)\right| & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \beta_{3,3}\left(\frac{13}{45}\right) & \text{if } x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

$$\beta_{3,2}d(\phi_3(c), S(\phi_2, \phi_2)) = \beta_{3,2}d\left(\phi_3(c), \frac{2\phi_2(c)}{5}\right)$$

$$= \begin{cases} \beta_{3,2}\left(\frac{3}{2}x^2\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \beta_{3,2}\left|\left(\frac{1}{9} - \frac{2}{5}x^2\right)\right| & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right] \\ \beta_{3,2}\left(\frac{1}{90}\right) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\beta_{3,3}d(\phi_3(c), S(\phi_3, \phi_3)) = \beta_{3,3}d\left(\phi_3(c), \frac{2\phi_3(c)}{5}\right)$$

$$= \beta_{3,3}\frac{3}{5}\phi_3(c)\begin{cases} \beta_{3,3}\left(\frac{3}{5}x^2\right) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ \beta_{3,3}\left(\frac{1}{15}\right) & \text{if } x \in \left[\frac{1}{3}, 1\right] \end{cases}$$

R.H.S of equation 2.7 is

$$= \alpha_1(1 - x^2) + 0 + \beta_{1,1}\left(\frac{3}{5}\right) + \beta_{1,2}\left(1 - \frac{2x^2}{5}\right) + \beta_{1,3}\left(1 - \frac{2x^2}{5}\right) + \beta_{2,1}\left(x^2 + \frac{2}{5}\right)$$

$$+ \beta_{2,2}\left(\frac{3}{5}x^2\right) + \beta_{2,3}\left(\frac{3}{5}x^2\right) + \beta_{3,1}\left(x^2 - \frac{2}{5}\right) + \beta_{3,2}\left(\frac{3}{5}x^2\right) + \beta_{3,3}\left(\frac{3}{5}x^2\right)$$

$$\text{if } x \in \left[0, \frac{1}{3}\right]$$

$$= \alpha_1(1 - x^2) + \alpha_2\left(x^2 - \frac{1}{9}\right) + \beta_{1,1}\left(\frac{3}{5}\right) + \beta_{1,2}\left(1 - \frac{2x^2}{5}\right) + \beta_{1,3}\left(\frac{43}{45}\right)$$

$$+ \beta_{2,1}\left(x^2 + \frac{2}{5}\right) + \beta_{2,2}\left(\frac{3}{5}x^2\right) + \beta_{2,3}\left(x^2 - \frac{2}{5}\right) + \beta_{3,1}\left(\frac{13}{45}\right) + \beta_{3,2}\left(\frac{1}{9} - \frac{2}{5}x^2\right)$$

$$+ \beta_{3,3}\left(\frac{1}{15}\right) \text{ if } x \in \left[\frac{1}{3}, \frac{1}{2}\right]$$

$$= \alpha_1\frac{3}{4} + \alpha_2\left(\frac{5}{36}\right) + \beta_{1,1}\left(\frac{3}{5}\right) + \beta_{1,2}\left(\frac{9}{10}\right) + \beta_{1,3}\left(\frac{43}{45}\right) + \beta_{2,1}\left(\frac{3}{20}\right) + \beta_{2,2}\left(\frac{3}{20}\right)$$

$$+ \beta_{2,3}\left(\frac{37}{180}\right) + \beta_{3,1}\left(\frac{13}{45}\right) + \beta_{3,2}\left(\frac{1}{90}\right) + \beta_{3,3}\left(\frac{1}{15}\right) \text{ if } x \in \left[\frac{1}{2}, 1\right]$$

which is

$$= \left(-\alpha_1 - \beta_{1,2}\frac{2}{5} - \beta_{1,3}\frac{2}{5} + \beta_{2,1} + \beta_{2,2} + \beta_{2,3}\frac{3}{5} + \beta_{3,1} + \beta_{3,2}\frac{3}{5} + \beta_{3,3}\frac{3}{5}\right)x^2$$

$$+\left(\alpha_1 + \beta_{1,1}\frac{3}{5} + \beta_{1,2} + \beta_{1,3} + \beta_{2,1}\frac{2}{5} - \beta_{3,1}\frac{2}{5}\right) \text{ if } x \in \left[0, \frac{1}{3}\right]$$

$$= \left(-\alpha_1 + \alpha_2 - \beta_{1,2}\frac{2}{5} + \beta_{2,1} + \beta_{2,2}\frac{3}{5} + \beta_{2,3} - \beta_{3,2}\frac{2}{5}\right)x^2 + \left(\alpha_1 - \alpha_2\frac{1}{9} + \beta_{1,1}\frac{3}{5}\right.$$

$$\left. + \beta_{1,3}\frac{43}{45} + \beta_{2,1}\frac{2}{5} - \beta_{2,3}\frac{2}{45} + \beta_{3,1}\frac{13}{45} + \beta_{3,2}\frac{1}{9} + \beta_{3,3}\frac{1}{15}\right) \text{ if } x \in \left[\frac{1}{3}, \frac{1}{2}\right]$$

$$= \alpha_1\frac{3}{4} + \alpha_2\left(\frac{5}{36}\right) + \beta_{1,1}\left(\frac{3}{5}\right) + \beta_{1,2}\left(\frac{9}{10}\right) + \beta_{1,3}\left(\frac{43}{45}\right) + \beta_{2,1}\left(\frac{3}{20}\right) + \beta_{2,2}\left(\frac{3}{20}\right)$$

$$+ \beta_{2,3}\left(\frac{37}{180}\right) + \beta_{3,1}\left(\frac{13}{45}\right) + \beta_{3,2}\left(\frac{1}{90}\right) + \beta_{3,3}\left(\frac{1}{15}\right) \text{ if } x \in \left[\frac{1}{2}, 1\right]$$

Now we take $\alpha_1 = \alpha_2 = \beta_{1,1} = \beta_{1,2} = \beta_{1,3} = \beta_{2,1} = \beta_{2,2} = \beta_{2,3} = \beta_{3,1} = \beta_{3,2} = \beta_{3,3} = C = 1/12$.

Hence R.H.S of equation 2.7 is

$$C\left(1 + \frac{3}{5} + 1 + 1 + \frac{2}{5} - \frac{2}{5}\right) + Cx^2\left(-1 - \frac{2}{5} - \frac{2}{5} + 1 + 1 + \frac{3}{5} + 1 + \frac{3}{5} + \frac{3}{5}\right)$$

$$= \frac{1}{12}\left(\frac{18}{5} + 3x^2\right) \text{ if } x \in \left[0, \frac{1}{3}\right]$$

$$= C\left(1 - \frac{1}{9} + \frac{3}{5} + \frac{43}{45} + \frac{2}{5} - \frac{2}{5} + \frac{13}{45} + \frac{1}{9} + \frac{1}{15}\right)$$

$$+ Cx^2\left(-1 + 1 - \frac{2}{5} + 1 + \frac{3}{5} + 1 - \frac{2}{5}\right) = \frac{1}{12}\left(\frac{147}{45} + \frac{9}{5}x^2\right) \text{ if } x \in \left[\frac{1}{3}, \frac{1}{2}\right]$$

$$= C\left(\frac{3}{4} + \frac{5}{36} + \frac{3}{5} + \frac{9}{10} + \frac{43}{45} + \frac{3}{20} + \frac{3}{20} + \frac{37}{180} + \frac{13}{45} + \frac{1}{90} + \frac{1}{15}\right) = C\left(\frac{759}{180}\right)$$

$$\text{if } x \in \left[\frac{1}{2}, 1\right]$$

Now, for all $x \in [0, 1]$, 2.7 holds. Hence $S$ is a Prešić-Hardy-Rogers contraction.

**Corollary 2.3.** "*Let* $(E, d)$ *be a complete metric space and* $I = [a, b]$ *be any closed interval in* $\mathbb{R}$. *Suppose* $E_0 = C(I, E)$ *denotes the set of all continuous function on* $I$ *to* $E, S : E_0^l \to E$ *is a Generalized Prešić contraction and* $\Omega_{\phi^*}$ *is a class of functions in* $E_0$, *which is topologically and algebraically closed. Then,* $S$ *has a unique PPF dependent fixed point in* $\Omega_{\phi^*}$."

**Proof.** For $\beta_{j,k} = \beta \; \forall \; j, k \in \{1, 2, \ldots, l, l+1\}$ with $j \neq k$ and $\beta_{j,j} = \beta_j \; \forall \; j \in \{1, 2, \ldots, l, l+1\}$, the Prešić-Hardy-Rogers contraction reduces into the generalized Prešić contraction.

**Corollary 2.4.** "*Let* $(E, d)$ *be a complete metric space and* $I = [a, b]$ *be any closed interval in* $\mathbb{R}$. *Suppose* $E_0 = C(I, E)$ *denotes the set of all continuous function on* $I$ *to* $E, S : E_0^l \to E$ *is a Prešić-Reich contraction and* $\Omega_{\phi^*}$ *is a class of functions in* $E_0$, *which is topologically and algebraically closed. Then,* $S$ *has a unique PPF dependent fixed point in* $\Omega_{\phi^*}$."

**Proof.** With $\beta = 0$, the generalized Prešić contraction reduces into the Prešić-Reich contraction.

**Corollary 2.5.** "*Let* $(E, d)$ *be a complete metric space and* $I = [a, b]$ *be any closed interval in* $\mathbb{R}$. *Suppose* $E_0 = C(I, E)$ *denotes the set of all continuous function on* $I$ *to* $E, S : E_0^l \to E$ *is a Prešić-Chatterjea contraction and* $\Omega_{\phi^*}$ *is a class of functions in* $E_0$, *which is topologically and algebraically closed. Then,* $S$ *has a unique PPF dependent fixed point in* $\Omega_{\phi^*}$."

**Proof.** With $\alpha_j = 0 \; \forall \; j \in \{1, 2, \ldots, l\}, \beta_j = 0 \; \forall \; j \in \{1, 2, \ldots, l, l+1\}$ and $\beta = \gamma$, the Prešić-Reich contraction reduces into the Prešić-Kannan contraction.

**Corollary 2.6.** "*Let* $(E, d)$ *be a complete metric space and* $I = [a, b]$ *be any closed interval in* $\mathbb{R}$. *Suppose* $E_0 = C(I, E)$ *denotes the set of all* *continuous function on I to* $E$, $S : E_0^l \to E$ *is a Prešić-Kannan contraction and* $\Omega_{\phi^*}$ *is a class of functions in* $E_0$, *which is topologically and algebraically closed. Then, S has a unique PPF dependent fixed point in* $\Omega_{\phi^*}$."

**Proof.** With $\alpha_j = 0 \ \forall\, j \in \{1, 2, \ldots, l\}$, the Prešić-Reich contraction reduces into the Prešić-Kannan contraction.

**Remark 2.7.** If we take $\beta_j = 0 \ \forall\, j \in \{1, 2, \ldots, l, l+1\}$, the Prešić-Reich contraction reduces into the Prešić contraction.

## 3. Conclusion

Inspired by the work of Bernfeld et al. [3] and Shukla et al. [21] we developed some PPF dependent fixed point results for nonself mapping in metric spaces for Prešić-Hardy-Rogers contraction, which is generalization of Prešić type contraction.

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