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On chromatic transversal domination in graphs

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Abstract

A proper k - coloring of a graph G is a function $f : V(G) \rightarrow \{1,2,...,k\}$ such that $f(u) \neq f(v)$ for all $uv \in E(G)$. The color class S_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k - coloring. A color class in a vertex coloring of a graph G is a subset of $V(G)$ containing all the vertices of the same color. The set $D \subseteq V(G)$ of vertices in a graph G is called dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. If $\mathscr{C} = \{S_1, S_2, ..., S_k\}$ is a k - coloring of a graph G then a subset D of $V(G)$ is called a transversal of $\mathscr{C}$ if $D \cap S_i \neq \phi$ for all $i \in \{1,2,...,k\}$. A dominating set D of a graph G is called a chromatic transversal dominating set (cdt - set) of G if D is transversal of every chromatic partition of G. Here we prove some characterizations and also investigate chromatic transversal domination number of some graphs.

Keywords

Coloring, Domination, Chromatic Transversal Dominating Set.

AMS Subject Classification

05C15, 05C69.

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Contents

1. Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$. We denote the degree of a vertex v in a graph G by $d_G(v)$. The maximum degree among the vertices of G is denoted by $\Delta(G)$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n.

The study of graph coloring and its related concepts are getting momentum due to its diversified applications for the solution of many real life problems such as scheduling time-table, compiler register allocation, assigning mobile and radio frequencies, etc. An excellent discussion on graph coloring is carried out by Zhang [11].

An independent set of vertices in a graph G is a set of pairwise non-adjacent vertices of G.

A proper k - coloring of a graph G is a function $f : V(G) \rightarrow \{1,2,...,k\}$ such that $f(u) \neq f(v)$ for all $uv \in E(G)$. The color class S_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k - coloring. Equivalently the chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of a graph G in such a way that no two adjacent vertices of G receive the same color. The minimum k such that we can partition $V(G) = S_1 \cup S_2 \cup ... \cup S_k$, where each S_i is independent set, is the chromatic number $\chi(G)$. A partition of $V(G)$ into $\chi(G)$ independent sets is called χ - partition of G.

The domination in graph is one of the fastest growing concepts in graph theory. Many variants of domination models are available in literature: Independent Domination [2, 6], Total Domination [3], Equitable Domination [7], Total Equitable Domination [1, 8, 9] are among worth to mention. Independent sets play a significant role in graph theory in general. They appear in theory of trees, coloring of graphs and matching theory.

The set $D \subseteq V(G)$ of vertices in a graph G is called dominating set if every vertex $v \in V(G)$ is either an element of D

or is adjacent to an element of D. The minimum cardinality of a dominating set is called the domination number of G which is denoted by $\gamma(G)$.

If $\mathscr{C} = \{S_1, S_2, ..., S_k\}$ is a k - coloring of a graph G then a subset D of $V(G)$ is called a transversal of $\mathscr{C}$ if $D \cap S_i \neq \phi$ for all $i \in \{1, 2, ..., k\}$. A dominating set D of a graph G is called a chromatic transversal dominating set (cdt - set) of G if D is transversal of every chromatic partition of G. The minimum cardinality of a cdt - set D of G is called the chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. This concept was introduced by Michaelraj *et al.* [5].

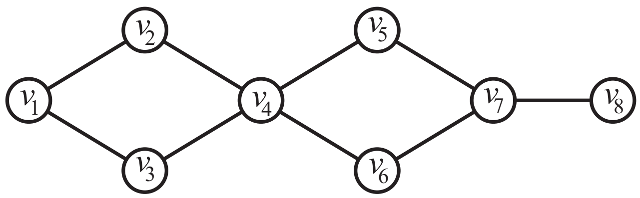

Figure 1. G

From the Figure 1, $\chi(G) = 2$ with the color partitions $S_1 = \{v_1, v_4, v_7\}$ and $S_2 = \{v_2, v_3, v_5, v_6, v_8\}$. The dominating set of G is $D = \{v_1, v_4, v_7\}$ with $|D| = 3$. But it is not chromatic transversal dominating set as $D \cap S_2 = \phi$. Moreover $D = \{v_1, v_4, v_7, v_8\}$ is a chromatic transversal dominating set of G with minimum cardinality because $D \cap S_i \neq \phi$.

Definition 1.1. *The square of a graph G denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance of 1 or 2 apart in G.*

Definition 1.2. *Let $G = (V(G), E(G))$ be a graph with $V(G) = V_1 \cup V_2 \cup V_3 \cup ... \cup V_t \cup T$, where each V_i is a set of all vertices having same degree with at least two elements and $T = V(G) \setminus \bigcup_{i=1}^{t} V_i$. The degree splitting $DS(G)$ is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining to each vertex of V_i for $1 \leq i \leq t$.*

Definition 1.3. *The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. We denote the resultant graph by $\widetilde{G}$.*

Here we contribute some characterizations and also investigate chromatic transversal domination number of some graph families.

For any graph theoretic notation and terminology we rely upon West [10]. For standard terminology and terms related to coloring are used in the sense of Zhang [11] while for any undefined terms related to the concept of domination we refer to Haynes *et al.* [4].

2. Main Results

Lemma 2.1. *For any graph G whose subgraph is K_3, the number of independent set is at least three.*

Proof: Let G be any graph whose subgraph is K_3. Then three mutually adjacent vertices give rise to three independent sets.

Lemma 2.2. *For any graph G, $\chi(G) \leq \gamma_{ct}(G)$.*

Proof: Let G be any graph with χ - partition is $\mathscr{C} = \{S_1, S_2, ..., S_{\chi(G)}\}$. Suppose D is a minimal dominating set of G with $D \cap S_i \neq \phi$ for all $i \in \{1, 2, ..., \chi(G)\}$. *i.e.* D is a minimal chromatic transversal dominating set of G. Then D has minimum $\chi(G)$ elements for dominate G with $D \cap S_i \neq \phi$ for all $i \in \{1, 2, ..., \chi(G)\}$. Therefore $\chi(G) \leq |D|$. Hence, $\chi(G) \leq \gamma_{ct}(G)$.

Theorem 2.3. *Let G be a graph with $\gamma(G) \leq \chi(G)$. Then $\gamma_{ct}(G) = \chi(G)$.*

Proof: Let G be a graph with $\gamma(G) \leq \chi(G)$. Let $(C) = \{S_1, S_2, ..., S_{\chi(G)}\}$ be χ - partition of G. If D_1 is a minimal dominating set of G with cardinality k for any $k \in \{1, 2, ..., \chi(G)\}$, then $|D_1| \leq \chi(G)$. Moreover $D_1 \cap S_j = \phi$ for all $j \in \{k+1, k+2, ..., \chi(G)\}$. Thus we required more $\chi(G) - k$ vertices for chromatic tranversal dominating set of G. Suppose D is a minimal chromatic transversal dominating set of G. Then $|D| = |D_1| + \chi(G) - k = \chi(G)$. Hence, $\gamma_{ct}(G) = \chi(G)$.

Lemma 2.4. $\chi(P_n^2) = 3$

Proof: Let $V(P_n) = V(P_n^2) = \{v_1, v_2, ..., v_n\}$ be the vertex set where $d_{P_n^2}(v_1) = d_{P_n^2}(v_n) = 2$, $d_{P_n^2}(v_2) = d_{P_n^2}(v_{n-1}) = 3$ and $d_{P_n^2}(v_i) = 4 = \Delta(G)$, for all $i \in \{3, 4, ..., n-2\}$.

Moreover by definition of P_n^2, K_3 is subgraph of P_n^2. Therefore number of independent sets are at least three.

Now we construct three independent sets of vertices as follows:

$S_1 = \{v_{3i+1} / 0 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}$,

$$S_2 = \begin{cases} \{v_{3i-1} / 1 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}; & \text{if } n \equiv 1 (\bmod 3) \\ \\ \{v_{3i+2} / 0 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}; & \text{otherwise} \end{cases}$$

And $S_3 = \{v_{3i} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\}$.

Further $V(P_n^2) = S_1 \cup S_2 \cup S_3$, where each S_i is independent. Hence $\chi(P_n^2) = 3$.

Theorem 2.5. $\gamma_{ct}(P_n^2) = \left\lceil \dfrac{n}{5} \right\rceil$.

Proof: If D is any color transversal dominating set of P_n^2 then v_3 must belongs to D as $d_{P_n^2}(v_3) = 4 = \Delta(P_n^2)$. From the Lemma 2.4,

$S_1 = \{v_{3i+1} / 0 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}$,

$$S_2 = \begin{cases} \{v_{3i-1} / 1 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}; & \text{if } n \equiv 1 (\bmod 3) \\ \\ \{v_{3i+2} / 0 \leq i \leq \lceil \frac{n}{3} \rceil - 1\}; & \text{otherwise} \end{cases}$$

And $S_3 = \{v_{3i} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor\}$ are three minimum independent sets of vertices with color 1, 2 and 3 respectively.

Now we construct a set D of vertices as follows:

$$D = \begin{cases} \{v_{5i+3}/0 \le i \le \lfloor \frac{n}{5} \rfloor\} & ;\text{for } n \equiv 3 \text{ or } 4(\bmod\ 5) \\ \{v_{5i+3}/0 \le i \le \frac{n}{5} - 1\} & ;\text{for } n \equiv 0(\bmod\ 5) \\ \{v_{5i+3}/0 \le i \le \lfloor \frac{n}{5} \rfloor - 1\} & ;\text{for } n \equiv 1 \text{ or } 2(\bmod\ 5) \end{cases}$$

Then $|D| = \lceil \frac{n}{5} \rceil$. Moreover D is a chromatic transversal dominating set of P_n^2 as $D \cap S_i \neq \phi$. Further we claim that $|D|$ is minimum because for any $u \in D$, $D - \{u\}$ is not a dominating set of P_n^2. Thus $D - \{u\}$ is not chromatic transversal dominating set of P_n^2. Therefore containing the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of P_n^2.

Hence, $\gamma_{ct}(P_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Lemma 2.6. $\chi(DS(P_n)) = 3$ *for* $n \ge 4$.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of P_n. For the graph $DS(P_n)$ added vertices are x and y and added edges are v_1y, v_ny and v_ix for $i = 2,3,4,...,n-1$. $|V(DS(P_n))| = n+2$ and $|E(DS(P_n))| = 2n-1$.

By definition of $DS(P_n)$, it is obvious K_3 is a subgraph of $DS(P_n)$. Therefore number of independent sets of $DS(P_n)$ are at least three.

Now we construct three independent sets of vertices as follows:
$S_1 = \{v_{2i+1}/0 \le i \le \lfloor \frac{n}{3} \rfloor\}$,
$S_2 = \{v_{2i}/1 \le i \le \lceil \frac{n}{3} \rceil\} \cup \{y\}$,
And $S_3 = \{x, v_n\}$

Further $V(DS(P_n)) = S_1 \cup S_2 \cup S_3$, where each S_i is independent. Hence $\chi(DS(P_n)) = 3$ for $n \ge 4$.

Lemma 2.7. $\gamma(DS(P_n)) = 2$ *for all* $n > 3$.

Proof: Let $DS(P_n)$ be the degree splitting graph of P_n. Now we consider the set of vertices $D = \{x, y\}$. Then $|D| = 2$. Moreover D is a dominating set of $DS(P_n)$ as $N[D] = V(DS(P_n))$. Further $|D|$ is minimum because for any $u \in D$, $D - \{u\}$ is not a dominating set of $DS(P_n)$. Hence $\gamma(DS(P_n)) = 2$.

Theorem 2.8. $\gamma_{ct}(DS(P_n)) = 3$ *for* $n \ge 4$.

Proof: Let $DS(P_n)$ be the degree splitting graph of P_n. From Lamma 2.6, $\chi(DS(P_n)) = 3$ and from Lemma 2.7, $\gamma(DS(P_n)) = 2$. Then $\gamma(DS(P_n)) < \chi(DS(P_n))$. Hence by Theorem 2.3, $\gamma_{ct}(DS(P_n)) = \chi(DS(P_n)) = 3$ for $n \ge 4$.

Lemma 2.9. $\chi(\widetilde{P_n}) = 3$.

Proof: Let P_n be path of n vertices. Vertices of degree one are known as terminal vertices and vertices of degree two are known as internal vertices. Let $\widetilde{P_n}$ be the graph obtained by switching of an arbitrary vertex v_i of P_n for $i = \{1,2,...,n\}$. Let $V(\widetilde{P_n}) = V(P_n) = \{v_1, v_2, ..., v_n\}$ be the vertex.

Moreover by definition of $\widetilde{P_n}$, K_3 is subgraph of $\widetilde{P_n}$. Therefore number of independent sets of $\widetilde{P_n}$ are at least three.

To prove this result we consider the following cases:
Case I: If either of the terminal vertex is switched.

Now we construct three independent sets of vertices as follows:
$S_1 = \{v_1\}$
$S_2 = \{v_{2i+1}/1 \le i \le \lceil \frac{n}{2} \rceil - 1\}$
And $S_3 = \{v_{2i}/1 \le i \le \lfloor \frac{n}{2} \rfloor\}$.

Further, $V(\widetilde{P_n}) = S_1 \cup S_2 \cup S_3$, where each S_i is an independent set of $\widetilde{P_n}$. Hence $\chi(\widetilde{P_n}) = 3$ if either of the terminal vertex is switched.

Case II: If either of the central vertex(vertices) is(are) switched.

Now we construct three independent sets of vertices as follows:

$$\left.\begin{aligned} S_1 &= \left\{v_{\frac{n+1}{2}}\right\} \\ S_2 &= \{v_{2i}/1 \le i \le \lfloor \tfrac{n}{2} \rfloor\} - \left\{v_{\frac{n+1}{2}}\right\} \\ S_3 &= \{v_{2i+1}/1 \le i \le \lfloor \tfrac{n}{2} \rfloor\} \end{aligned}\right\} \text{For } n \text{ is odd.}$$

And

$$\left.\begin{aligned} S_1 &= \left\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\right\} \\ S_2 &= \{v_{2i+1}/0 \le i \le \tfrac{n}{2} - 1\} - V_1 = \left\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\right\} \\ S_3 &= \{v_{2i}/1 \le i \le \lfloor \tfrac{n}{2} \rfloor\} - V_1 = \left\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\right\} \end{aligned}\right\} \text{For } n \text{ is even.}$$

Further, $V(\widetilde{P_n}) = S_1 \cup S_2 \cup S_3$, where each S_i is an independent set of $\widetilde{P_n}$. Hence $\chi(\widetilde{P_n}) = 3$ if either of the central vertex(vertices) is(are) switched.

Lemma 2.10.

$$\gamma(\widetilde{P_n}) = \begin{cases} 2; & \textit{if either of the terminal vertex is switched} \\ 3; & \textit{if either of the internal vertex is switched} \end{cases}$$

Proof: Let $\widetilde{P_n}$ be the graph obtained by switching of an arbitrary vertex v_i of P_n for $i \in \{1,2,3,...,n\}$. To prove this result we consider the following cases:
Case I: If either of the terminal vertex is switched

Without loss of generality, we switched a vertex v_1. We consider the set of vertices $D = \{v_1, v_3\}$. Then $|D| = 2$. Moreover D is a dominating set of $\widetilde{P_n}$ as $N[D] = V(\widetilde{P_n})$. Further $|D|$ is minimum because for any $u \in D$, $D - \{u\}$ is not a dominating set of $\widetilde{P_n}$. Hence $\gamma(\widetilde{P_n}) = 2$.

Case II: If either of the internal vertex is switched

We switched a vertex v_i for any $i \in \{3,4,...,n-2\}$. We consider the set of vertices $D = \{v_{i-2}, v_i, v_{i+2}/3 \le i \le n_2\}$. Then $|D| = 3$. Moreover D is a dominating set of $\widetilde{P_n}$ as $N[D] = V(\widetilde{P_n})$. Further $|D|$ is minimum because for any $u \in D$, $D - \{u\}$ is not a dominating set of $\widetilde{P_n}$. Hence $\gamma(\widetilde{P_n}) = 3$.

Theorem 2.11. $\gamma_{ct}(\widetilde{P_n}) = 3$ *for* $n \ge 3$.

Proof: Let $\widetilde{P_n}$ be the graph obtained by switching of an arbitrary vertex v_i of P_n for $i \in \{1,2,3,...,n\}$. From Lamma 2.9, $\chi(\widetilde{P_n}) = 3$ and from Lemma 2.10, $\gamma(\widetilde{P_n}) \le \chi(\widetilde{P_n})$. Hence by Theorem 2.3, $\gamma_{ct}(\widetilde{P_n}) = \chi(\widetilde{P_n}) = 3$ for $n \ge 3$.

3. Conclusion

The concept of chromatic transversal domination relates two important concepts of graph theory namely, coloring and domination. We have proved some characterizations and obtained chromatic transversal domination number of some graph families.

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