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Fixed point theorems with PPF dependence in strong partial b-metric spaces

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Abstract. In this study, PPF dependent fixed point theorems are proved for a nonlinear operator, where the domain space $C[[a,b],E]$ is distinct from the range space, E, which is a Strong Partial b-metric space (SPbMS). We obtain existence and uniqueness of PPF dependent fixed point results for the defined mappings under SPbMS. Our results are the extension of fixed point results in SPbMS. Examples are provided in the support of results.

Keywords: PPF dependent fixed point; uniqueness; strong partial b-metric space.

1 Introduction

Fixed point theory has emerged as a highly useful tool in the study of nonlinear processes during the last few decades. Fixed point concepts and findings in pure and applied analysis, topology, and geometry have been developed in particular. The well-known Banach contraction principle [21] is a key of this theory. The works of Bourbaki [14] and Bakhtin [7] influenced the concept of b-metric. In 1993, Czerwik [22] provided a weaker assumption than the triangle inequality and explicitly defined a b-metric space in order to generalise the Banach contraction mapping theorem. Matthews [23], in 1994 proposed the concept of partial metric space as part of the research of denotational semantics of dataflow networks and demonstrated how the Banach contraction principle may be adapted to the partial metric context for programme verification applications. In [24], the notion of SPbMSs was introduced. They also discussed the relationship between strong b-metric and SPbMSs.

While referring to a fixed point for mappings with different domains and ranges, Bernfeld et al. [25] originally used the terms "PPF dependent fixed point"

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or "fixed point with PPF dependence" in 1977. Additionally, they created the concept of Banach type contraction and showed some important outcomes under this. For various contraction mappings, recent research has shown the existence and uniqueness of PPF dependent fixed points [20],[8],[15],[2],[9], and those interested in the applications can find PPF dependent solutions to periodic boundary value problems and functional differential equations that may depend on past, present, and future considerations [3],[4],[26]]. Wardowski [5] first discussed the F-contraction, a new contraction, in 2012. Under this contraction, he developed more fixed point observations and created a fixed point theorem. Following that, Abbas et al. [10] provided an extension of the idea of F-contraction to reach specific fixed point findings. A striking generalisation of F-contraction on graphs was described by Batra et al. [18], [19]. Acar et al. [16],[17] revealed the development of a generalised multi-valued F-contraction mapping to explore fixed point theory findings in a complete metric space. Wardowski [6] investigated an extension of the Banach fixed point theorem in a new class of contraction mappings on metric spaces known as (ϕ, F)-contraction (nonlinear F-contraction) in 2018. In [24], the notion of Strong Partial b-Metric Spaces (SPbMSs) was introduced. They also discussed the relationship between strong b-metric and SPbMSs.

Inspired by the outcomes of Kari et al. [1], Wardowski [5] and Bernfeld et.al. [25] we establish some results of PPF dependent fixed point for nonlinear F-contraction type mappings in the case of SPbMSs.

2 Preliminaries

Here, we provide the relevant definitions and findings for different spaces and different type of contractions that will be helpful for further explanation.

Definition 1. [25] "A function $\psi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of a nonself mapping S if $S\psi = \psi(c)$ for some $c \in I$."

Definition 2. [24] "A map $d : E \times E \to \mathbb{R}_0^+$ is a strong partial b-metric on a non empty set E if for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$ and $\alpha \geq 1$ the following conditions hold:

(SPbM1) $\mathfrak{a} = \mathfrak{b} \Leftrightarrow d(\mathfrak{a}, \mathfrak{a}) = d(\mathfrak{b}, \mathfrak{b}) = d(\mathfrak{a}, \mathfrak{b})$;

(SPbM2) $d(\mathfrak{a}, \mathfrak{a}) \leq d(\mathfrak{a}, \mathfrak{b})$;

(SPbM3) $d(\mathfrak{a}, \mathfrak{b}) = d(\mathfrak{b}, \mathfrak{a})$;

(SPbM4) $d(\mathfrak{a}, \mathfrak{b}) \leq d(\mathfrak{a}, \mathfrak{c}) + \alpha d(\mathfrak{c}, \mathfrak{b}) - d(\mathfrak{c}, \mathfrak{c})$.

The triple (E, d, α) is called a Strong Partial b-Metric Space (SPbM)."

Definition 3. [24] " Let (E, d, α) be a SPbMS. Then

1. A sequence $\{\mathfrak{a}_n\}$ in (E, d, α) converges to a point $\mathfrak{a} \in E$ if $d(\mathfrak{a}, \mathfrak{a}) = \lim_n d(\mathfrak{a}_n, \mathfrak{a}) = \lim_n d(\mathfrak{a}_n, \mathfrak{a}_n)$.

2. A sequence $\{\mathfrak{a}_n\}$ in (E, d, α) is Cauchy if the $\lim_{n,m} d(\mathfrak{a}_n, \mathfrak{a}_m)$ exists and finite."

Definition 4. [5] "Let $\mathfrak{F}$ be the family of all continuous functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(F1) F is strictly increasing;

(F2) For each sequence $\{a_n\} \in \mathbb{N}$ of positive numbers

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty; \tag{2.1}$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $S : E \times E$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $\psi, \xi \in E_0$

$$(d(S\psi, S\xi)) > 0 \implies \tau + F(d(S\psi, S\xi)) \leq F(d(\psi(c), \xi(c))).\text{"} \tag{2.2}$$

Turinici [13] observed that the condition (F2) can be relaxed to the form
(F2') $\lim_{n\to\infty} F(a_n) = -\infty$.

Definition 5. [6] "A mapping $S : E \to E$ is said to be a (ϕ, F)-contraction (or nonlinear F-contraction) if there exist the functions $F : (0, \infty) \to \mathbb{R}$ and $\phi : (0, \infty) \to (0, \infty)$ satisfy the following

(H1) F satisfies (F1) and (F2');

(H2) $\liminf_{s\to t^+} \phi(s) > 0$ for all $t \geq 0$;

(H3) $\phi(d^{'}(\psi, \xi)) + F(d(S\psi, S\xi)) \leq F(d^{'}(\psi, \xi))$ for all $\psi, \xi \in E$ such that $S\psi \neq S\xi$."

Theorem 2.1. *[6] "Let (E, d) be a complete metric space and $S : E_0 \to E$ be a (ϕ, F)-contraction. Then S has a unique fixed point."*

3 Main Results

Through the paper,(E, d) is the complete SPbMS. S is an operator from E_0 to E, where $E_0 = C[[a, b], E]$, is the collection of all continuos fuctions from $[a, b]$ to E. $\mathfrak{F}$ is a family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfies (F1), (F2), (F2'), (F3). $\mathbb{R}$ represents the set of real numbers and $\mathbb{N}$ is the set of natural numbers. Φ is the family of all functions

$\phi : (0, \infty) \to (0, \infty)$ with the condition $\liminf_{s \to t^+} \phi(s) > 0$ for all $t \geq 0$.

Let $\mathcal{F}$ be the class of functions which satisfy

$$\mathcal{F} = f : (0, \infty) \to [0, \frac{1}{2}) : f(z_n) \to \frac{1}{2} \implies z_n \to 0 \text{ as } n \to \infty.$$

Theorem 3.1. Let (E, d, α) be a complete SPbMS and $S : E_0 \to E$ be a non-self map. Suppose, $\exists$ $f \in \mathcal{F}$ such that for all $\psi, \xi \in E_0$ and for some $c \in I$, with $\psi(c) \neq \xi(c)$,

$$d(S\psi, S\xi) \leq f(d^{'}(\psi, \xi))\{d(\psi(c), S\psi) + d(\xi(c), S\xi)\}. \tag{3.1}$$

Then, S has a unique PPF dependent fixed point in E_0.

Proof. Let $\psi_0 \in E_0$ be any arbitrary fuction, so as $S\psi_0 = y_1$ for any $y_1 \in E$. We choose $\psi_1 \in E_0$ such as $y_1 = \psi_1(c)$ and $d^{'}(\psi_0, \psi_1) = d(\psi_0(c), \psi_1(c)$. Similary, suppose $\psi_1 \in E_0$, we choose $\psi_2 \in E_0$ such as $y_2 = \psi_2(c) = S\psi_1$. On the same step, we obtain a sequence $\{\psi_n\}$ in E_0,

$$\psi_{n+1}(c) = S\psi_n = y_{n+1} \text{ for } c \in I \text{ and } \forall \ n \geq 0.$$

Suppose, there exists $n \geq 0$ such that $\psi_{n+1}(c) = S\psi_n$, then obviously ψ_n is a PPF dependent fixed point of S. So, assume $\psi_{n+1}(c) \neq S\psi_n$ $\forall$ $n \geq 0$ and $c \in I$.
Now, define $D_n = d^{'}(\psi_{n+1}, \psi_n) = d(\psi_{n+1}(c), \psi_n(c))$ $\forall$ $n \geq 0$. By inequality (3.1), we get

$$\begin{aligned}
D_{n+1} &= d^{'}(\psi_{n+2}, \psi_{n+1}) \\
&= d(\psi_{n+2}(c), \psi_{n+1}(c)) \\
&= d(S\psi_{n+1}, S\psi_n) \\
&\leq f(d^{'}(\psi_{n+1}, \psi_n))\{d(\psi_{n+1}(c), S\psi_{n+1}) + d(\psi_n(c), S\psi_n\} \\
&< \frac{1}{2}\{d(\psi_{n+1}(c), S\psi_{n+1}) + d(\psi_n(c), S\psi_n\} \\
&= \frac{1}{2}\{d(\psi_{n+1}(c), \psi_{n+2}(c)) + d(\psi_n(c), \psi_{n+1}(c)\} \\
&= \frac{1}{2}\{D_n + D_{n+1}\}.
\end{aligned}$$

Clearly, $D_{n+1} < D_n$ $\forall$ $n \geq 0$. Hence, $\{D_n\}$ is a monotonically decreasing and bounded below sequence. So, $\exists$ $\beta \geq 0$ so as

$$\lim_{n \to \infty} D_n = \beta.$$

Now, assume $\beta > 0$. Then, by inequality (3.1), we get

$$d^{'}(\psi_{n+2}, \psi_{n+1}) \leq f(d^{'}(\psi_{n+1}, \psi_n))\{d^{'}(\psi_{n+1}, \psi_{n+2}) + d^{'}(\psi_n, \psi_{n+1})\};$$

that is

$$D_{n+1} \leq f(D_n)\{D_{n+1} + D_n\}.$$

That implies
$$\frac{D_{n+1}}{D_{n+1}+D_n} \leq f(D_n) \quad \forall \;\; n \geq 0.$$
Applying $n \to \infty$, we get $\frac{1}{2} \leq \lim_{n\to\infty} f(D_n)$, but $\frac{1}{2} > \lim_{n\to\infty} f(D_n)$, because $f \in \mathcal{F}$. Which is a contradiction. So, $\lim_{n\to\infty}(D_n) = \beta = 0$.

We demonstrate that $\{S\psi_n\}$ is a Cauchy sequence in E. Let $m < n$. So, by inequality (3.1), we get
$$\begin{aligned} d(S\psi_{m+1}, S\psi_{n+1}) &\leq f(d^{'}(\psi_m, \psi_n))\{d(\psi_m(c), S\psi_m) + d(\psi_n(c), S\psi_n)\} \\ &\leq \frac{1}{2}\{d(\psi_m(c), \psi_{m+1}(c)) + d(\psi_n(c), \psi_{n+1}(c))\}. \end{aligned}$$
As $m, n \to \infty, d(\psi_m(c), \psi_{m+1}(c))$ and $d(\psi_n(c), \psi_{n+1}(c)) \to 0$. So,
$$d(S\psi_{m+1}, S\psi_{n+1}) \to 0 \;\; \text{as} \;\; n \to \infty.$$
So, $\{S\psi_n\}$ is a cauchy sequence. Now, $\{S\psi_n\}$ is a Cauchy sequence and by hypothesis E is complete. So, $\exists \;\; \psi^* \in E_0$ such that
$$\lim_{n\to\infty} S\psi_n = \psi^*(c).$$
Now, by (SPbM4)
$$\begin{aligned} d(S\psi^*, \psi^*(c)) &\leq d(S\psi^*, S\psi_n) + \alpha d(S\psi_n, \psi^*(c)) - d(S\psi_n, S\psi_n) \\ &\leq f(d^{'}(\psi^*, \psi_n))\{d(\psi^*(c), S\psi^*) + d(\psi_n(c), S\psi_n)\} + \alpha d(S\psi_n, \psi^*(c)) - d^{'}(\psi_{n+1}, \psi_{n+1}) \\ &\leq f(d^{'}(\psi^*, \psi_n))\{d(\psi^*(c), S\psi^*) + d(\psi_n(c), S\psi_n)\} + \alpha d(S\psi_n, \psi^*(c)). \end{aligned}$$
So,
$$d(S\psi^*, \psi^*(c))(1 - f(d^{'}(\psi^*, \psi_n))) \leq f(d^{'}(\psi^*, \psi_n))d(\psi_n(c), S\psi_n) + \alpha d(\psi_{n+1}(c), \psi^*(c)),$$
which implies
$$d(S\psi^*, \psi^*(c)) \leq \frac{f(d^{'}(\psi^*, \psi_n))}{1 - f(d^{'}(\psi^*, \psi_n))} D_n + \frac{\alpha}{1 - f(d^{'}(\psi^*, \psi_n))} d(\psi_{n+1}(c), \psi^*(c)) \tag{3.2}$$
As $n \to \infty$ right hand side of (3.2) is zero. So,
$$d(S\psi^*, \psi^*(c)) = 0. \tag{3.3}$$
Now, by (SPbM2) $d(S\psi^*, S\psi^*) \leq d(S\psi^*, \psi^*(c))$.
Since, $S : E \times E \to [0, \infty)$ and $d(S\psi^*, \psi^*(c)) = 0$.
So, $d(S\psi^*, S\psi^*) = 0$.
Similarly, we can show that $d(\psi^*(c), \psi^*(c)) = 0$. Thus, we get $d(\psi^*(c), \psi^*(c)) = d(S\psi^*, \psi^*(c)) = d(S\psi^*, S\psi^*)$. So, by (SPbMS1) $S\psi^* = \psi^*(c)$. Hence $\psi^* \in E_0$ is a PPF dependent fixed point of S.

Uniqueness: If possible, let ξ^* be any other PPF dependent fixed point of S. So, $S\xi^* = \xi^*(c)$.

Using inequality (3.1), we get
$$\begin{aligned} d^{'}(\psi^*, \xi^*) &= d(S\psi^*, S\xi^*) \\ &\leq f(d^{'}(\psi^*, \xi^*))\{d(\psi^*(c), S\psi^*) + d(\xi,^*(c), S\xi^*)\}. \end{aligned}$$

By using equation (3.3), we have $d^{'}(\psi^{*},\xi^{*})=0$.
Now, $d^{'}(\psi^{*},\psi^{*})=d^{'}(\xi,^{*},\xi^{*})=0$. $[\because d^{'}(\psi^{*},\psi^{*})\leq d^{'}(\psi^{*},\xi^{*})$ and $d^{'}(\xi,^{*},\xi^{*})\leq d^{'}(\psi^{*},\xi^{*})$.]
So, $d^{'}(\psi^{*},\psi^{*})=d^{'}(\psi^{*},\xi^{*})=d^{'}(\xi,^{*},\xi^{*})$. Hence $\psi^{*}=\xi^{*}$. Thus, S has unique PPF dependent fixed point $\psi^{*}\in E_0$. □

Corollary 3.2. "Let (E,d,α) be a complete strong b-metric space and $S:E_0\to E$ be a non-self map. Suppose, $\exists\ f\in\mathcal{F}$ such that for all $\psi,\xi\in E_0$ with $\psi(c)\neq\xi(c)$, where $c\in I$,

$$d(S\psi(c),S\xi)\leq f(d^{'}(\psi,\xi))\{d(\psi(c),S\psi)+d(\xi(c),,S\xi)\}.$$

Then, S has a unique PPF dependent fixed point in E_0."

Corollary 3.3. "Let (E,d,α) be a complete metric space and $S:E_0\to E$ be a non-self map. Suppose, $\exists\ f\in\mathcal{F}$ such that for all $\psi,\xi\in E_0$ with $\psi(c)\neq\xi(c)$, where $c\in I$,

$$d(S\psi(c),S\xi)\leq f(d^{'}(\psi,\xi))\{d(\psi(c),S\psi)+d(\xi(c),S\xi)\}.$$

Then, S has a unique PPF dependent fixed point in E_0."

Example 1. Consider

$$S(h)=\frac{5}{9}h\left(\frac{1}{3}\right)+\frac{1}{81},\ \text{for every h}\in E_0$$

$$\psi(x)=\begin{cases}\frac{1}{2}-x^2 \text{ if } x\in[0,\frac{1}{2}]\\ \frac{1}{4} \text{ if } x\in[\frac{1}{2},1]\end{cases}$$

$$\xi(x)=\begin{cases}x \text{ if } x\in[0,\frac{1}{3}]\\ \frac{1}{3} \text{ if } x\in[\frac{1}{3},1]\end{cases}$$

Clearly, $\psi\left(\frac{1}{3}\right)=\frac{7}{18}$ and $\xi\left(\frac{1}{3}\right)=\frac{1}{3}$, i.e. $\psi\left(\frac{1}{3}\right)\neq\xi\left(\frac{1}{3}\right)$.
Now, define $d(x,y)=|x-y|$, $d'(\psi,\xi)=|\psi-\xi|$ and $f(z_n)=\frac{1}{2}-z_n$, where $z_n=\frac{1}{n},n\in\mathbb{N}$. We find

$$S\psi=\frac{5}{9}\psi\left(\frac{1}{3}\right)+\frac{1}{81}=\frac{37}{162},$$

$$S\xi=\frac{5}{9}\xi\left(\frac{1}{3}\right)+\frac{1}{81}=\frac{16}{81}.$$

From condition (3.1), we calculated

$$d(S\psi,S\xi)=|S\psi-S\xi|=\frac{5}{162}$$

and

$$f(d^{'}(\psi,\xi))\{d(\psi(c),S\psi)+d(\xi(c),S\xi)\} = \frac{1}{2}\left\{\left|\frac{43}{162}-\frac{7}{18}\right|+\left|\frac{19}{81}-\frac{1}{3}\right|\right\}$$
$$= \frac{18}{162}$$

which satisfies the condition (3.1). Thus, S has ξ as unique PPF dependent fixed point, as $S\xi = \xi(c)$ for $c = \frac{19}{81} \in I$.

Now, we consider $\mathcal{G}$ is the class of functions which satisfy

$$\mathcal{G} = g : (0,\infty) \to [0,\frac{1}{3}) : g(z_n) \to \frac{1}{3} \implies z_n \to 0 \text{ as } n \to \infty.$$

Theorem 3.4. Let (E, d, α) be a complete SPbMS and $S : E_0 \to E$ be a self map. Assume, $\exists\ g \in \mathcal{G}$ such that $\forall\ \psi,\xi \in E_0$, with $\psi(c) \neq \xi(c)$, for each $c \in I$,

$$d(S\psi, S\xi) \leq g(d^{'}(\psi,\xi))\{d(\psi(c),S\psi)+d(\xi(c),S\xi)+d^{'}(\psi,\xi)\}. \tag{3.4}$$

Then, S has a unique PPF dependent fixed point in E_0.

Proof. Let $\psi_0 \in E_0$ be any arbitrary fuction, so as $S\psi_0 = y_1$ for any $y_1 \in E$. We choose $\psi_1 \in E_0$ such as $y_1 = \psi_1(c)$ and $d^{'}(\psi_0,\psi_1) = d(\psi_0(c),\psi_1(c)$. Similary, suppose $\psi_1 \in E_0$, we choose $\psi_2 \in E_0$ such as $y_2 = \psi_2(c) = S\psi_1$. On the same step, we obtain a sequence $\{\psi_n\}$ in E_0,

$$\psi_{n+1}(c) = S\psi_n = y_{n+1} \text{ for } c \in I \text{ and } \forall\ n \geq 0.$$

Suppose, there exists $n \geq 0$ such that $\psi_{n+1}(c) = S\psi_n$, then obviously ψ_n is a PPF dependent fixed point of S. So, assume $\psi_{n+1}(c) \neq S\psi_n\ \forall\ n \geq 0$ and $c \in I$.
Now, define $D_n = d^{'}(\psi_{n+1},\psi_n) = d^{(}\psi_{n+1}(c),\psi_n(c))\ \forall\ n \geq 0$. By inequality (3.4), we get

$$\begin{aligned}
D_{n+1} &= d^{'}(\psi_{n+2},\psi_{n+1}) \\
&= d(\psi_{n+2}(c),\psi_{n+1}(c)) \\
&= d(S\psi_{n+1},S\psi_n) \\
&\leq g(d^{'}(\psi_{n+1},\psi_n))\{d(\psi_{n+1}(c),S\psi_{n+1})+d(\psi_n(c),S\psi_n)+d^{'}(\psi_{n+1},\psi_n)\} \\
&< \frac{1}{3}\{d(\psi_{n+1}(c),\psi_{n+2}(c))+d(\psi_n(c),\psi_{n+1}(c))+d^{'}(\psi_n,\psi_{n+1})\} \\
&= \frac{1}{3}\{2D_n + D_{n+1}\}.
\end{aligned}$$

Clearly, $D_{n+1} < D_n\ \forall\ n \geq 0$. Hence, $\{D_n\}$ is a monotonically decreasing and bounded below sequence. So, $\exists\ \beta \geq 0$ so as

$$\lim_{n\to\infty} D_n = \beta.$$

Now, suppose $\beta > 0$. Then, by inequality (3.4), we have
$d(S\psi_{n+1}, S\psi_n) \leq g(d^{'}(\psi_{n+1}, \psi_n))\{d(\psi_{n+1}(c), S\psi_{n+1}) + d(\psi_n(c), S\psi_n) + d^{'}(\psi_{n+1}, \psi_n)\}$;
that is

$$D_{n+1} \leq g(D_n)\{D_{n+1} + 2D_n\}.$$

That implies

$$\frac{D_{n+1}}{D_{n+1} + 2D_n} \leq g(D_n) \quad \forall \quad n \geq 0.$$

Applying $n \to \infty$, we get $\frac{1}{3} \leq \lim_{n\to\infty} g(D_n)$, but $\frac{1}{3} > \lim_{n\to\infty} g(D_n)$, because $g \in \mathcal{G}$. Which is a contradiction. So, $\lim_{n\to\infty}(D_n) = \beta = 0$.

We demonstrate that $\{S\psi_n\}$ is a Cauchy sequence in E. Let $m < n$. So, by (SpBM4) and inequality (3.4), we have

$$\begin{aligned}
d(S\psi_m, S\psi_n) &\leq g(d^{'}(\psi_m, \psi_n))\{d(\psi_m(c), S\psi_m) + d(\psi_n(c), S\psi_n) + d(\psi_m(c), \psi_n(c))\}\\
&\leq \frac{1}{3}\{d(\psi_m(c), S\psi_m) + d(\psi_n(c), S\psi_n) + d^{(}\psi_m(c), \psi_{n+1}(c)) + \alpha d(\psi_{n+1}(c), \psi_n(c))\\
&\quad - d(\psi_{n+1}(c), \psi_{n+1}(c))\}\\
&\leq \frac{1}{3}\{d(\psi_m(c), \psi_{m+1}(c)) + d(\psi_n(c), \psi_{n+1}(c)) + d(\psi_m(c), \psi_{n+1}(c)) + \alpha d(\psi_{n+1}(c), \psi_n(c))\}\\
&\leq \frac{1}{3}\{d(\psi_m(c), \psi_{m+1}(c)) + \alpha d(\psi_m(c), \psi_{m+1}(c)) + d(\psi_{m+1}(c), \psi_{n+1}(c))\\
&\quad - d(\psi_{m+1}(c), \psi_{m+1}(c)) + (1+\alpha)d(\psi_{n+1}(c), \psi_n(c))\}\\
&\leq \frac{1}{3}\{d(\psi_m(c), \psi_{m+1}(c)) + \alpha d(\psi_m(c), \psi_{m+1}(c)) + d(\psi_{m+1}(c), \psi_{n+1}(c))\\
&\qquad + (1+\alpha)d(\psi_{n+1}(c), \psi_n(c))\},
\end{aligned}$$

which means

$$d(\psi_{m+1}(c), \psi_{n+1}(c)) \leq \frac{\alpha+1}{2}\{D_m + D_n\}.$$

As $m, n \to \infty, d(\psi_m(c), \psi_{m+1}(c))$ and $d(\psi_n, \psi_{n+1}) \to 0$.
So, $d(\psi_{m+1}(c), \psi_{n+1}(c)) \to 0$ as $n \to \infty$.
Hence, $\{\psi_n(c)\}$ is a Cauchy sequence. We can say that $\{S\psi_n\}$ is a Cauchy sequence. Now, by hypothesis, E is complete. So, $\exists \ \xi^* \in E$ so that

$$\lim_{n\to\infty} S^n\psi_0 = \xi^*(c).$$

Now, by (SPbMS4) and inequation (3.4),

$$\begin{aligned}
d(S\xi^*, \xi^*(c)) &\leq d(S\xi^*, S\psi_n) + \alpha d(S\psi_n, \xi^*(c)) - d(S\psi_n, S\psi_n)\\
&\leq g(d^{'}(\xi^*, \psi_n))\{d(\xi^*(c), S\xi^*) + d(\psi_n(c), S\psi_n) + d(\xi^*(c), \psi_n(c))\} + \alpha d(\psi_{n+1}(c), \xi^*(c))\\
&\quad - d(\psi_{n+1}(c), \psi_{n+1}(c))\\
&\leq g(d^{'}(\xi^*, \psi_n))\{d(\xi^*(c), S\xi^*) + d(\psi_n(c), S\psi_n) + d(\xi^*(c), \psi_n(c))\} + \alpha d(\psi_{n+1}(c), \xi^*(c)).
\end{aligned}$$

So,
$d(S\xi^*, \xi^*(c))(1 - g(d^{'}(\xi^*, \psi_n))) \leq g(d^{'}(\xi^*, \psi_n))d(\psi_n(c), S\psi_n) + g(d^{'}(\xi^*, \psi_n))d(\psi_n(c), \xi^*(c))$

$$+\alpha d(\psi_{n+1}(c),\xi^*(c)).$$

That is

$$d(S\xi^*,\xi^*(c)) \le \frac{g(d'(\xi^*,\psi_n))}{1-g(d'(\xi^*,\psi_n))}d(\psi_n(c),\psi_{n+1}(c)) + \frac{g(d'(\xi^*,\psi_n))}{1-g(d'(\xi^*,\psi_n))}d(\psi_n(c),\xi^*(c))$$
$$+\frac{\alpha}{1-g(d'(\xi^*,\psi_n))}d(\psi_{n+1}(c),\xi^*(c)).$$

This implies

$$d(S\xi^*,\xi^*(c)) \le \frac{g(d'(\xi^*,\psi_n))}{1-g(d'(\xi^*,\psi_n))}D_n + \frac{g(d'(\xi^*,\psi_n))}{1-g(d'(\xi^*,\psi_n))}d(\psi_n(c),\xi^*(c)) \tag{3.5}$$
$$+\frac{\alpha}{1-g(d'(\xi^*,\psi_n))}d(\psi_{n+1}(c),\xi^*(c)). \tag{3.6}$$

As $n \to \infty$, right hand side of (3.5) is zero. So,

$$d(S\xi^*,\xi^*(c)) = 0. \tag{3.7}$$

Now, by (SPbMS2) $d(S\xi^*,S\xi^*) \le d(S\xi^*,\xi^*(c))$.
Since, $S : E \times E \to [0,\infty)$ and $d(S\xi^*,\xi^*(c)) = 0$. So, $d(S\xi^*,S\xi^*) = 0$. Similarly, we can show that $d(\xi^*(c),\xi^*(c)) = 0$. Thus, we get $d(\xi^*(c),\xi^*(c)) = d(S\xi^*,\xi^*(c)) = d(S\xi^*,S\xi^*)$. So, by (SPbMS1) $S\xi^* = \xi^*(c)$. Hence $\xi^* \in E$ is a PPF dependent fixed point of S.
Uniqueness: Let if possible ψ^* is another fixed point of S. So, $S\psi^* = \psi^*(c)$.
Using inequality (3.4), we get

$$\begin{aligned} d'(\xi^*,\psi^*) &= d(\xi^*(c),\psi^*(c)) = d(S\xi^*,S\psi^*) \\ &\le g(d'(\xi^*,\psi^*))\{d(\xi^*(c),S\xi^*) + d(\psi^*(c),S\psi^*) + d'(\xi^*,\psi^*)\} \\ &\le \frac{1}{3}\{d(\xi^*(c),S\xi^*) + d(\psi^*(c),S\psi^*) + d'(\xi^*,\psi^*)\}. \end{aligned}$$

and

$$\frac{2}{3}d'(\xi^*,\psi^*) \le \frac{1}{3}\{d(\xi^*(c),S\xi^*) + d(\psi^*(c),S\psi^*).$$

By using equation (3.7), we have $d'(\xi^*,\psi^*) = 0$.
Now, $d'(\xi^*,\xi^*) = d'(\psi^*,\psi^*) = 0$. $[\because d'(\xi^*,\xi^*) \le d'(\xi^*,\psi^*)$ and $d'(\psi^*,\psi^*) \le d'(\xi^*,\psi^*).]$
So, $d'(\xi^*,\xi^*) = d'(\xi^*,\psi^*) = d'(\psi^*,\psi^*)$. Hence $\xi^* = \psi^*$. Thus, S has exactly one PPF depenent fixed point $\xi^* \in E$. □

Corollary 3.5. "Let (E,d,α) be a complete strong b-metric space and $S : E_0 \to E$ be a non-self map. Suppose, $\exists\ g \in \mathcal{G}$ such that for all $\psi,\xi \in E$ with $\psi(c) \ne \xi(c)$, for each $c \in I$,

$$d(S\psi,S\xi) \le g(d'(\psi,\xi))\{d(\psi(c),S\psi) + d(\xi(c),S\xi) + d'(\psi,\xi)\}.$$

Then, S has a unique PPF dependent fixed point in E_0."

Corollary 3.6. "Let (E,d,α) be a complete metric space and $S : E_0 \to E$ be a non-self map. Suppose, $\exists\ g \in \mathcal{G}$ such that for all $\psi,\xi \in E_0$ with $\psi(c) \ne \xi(c)$ for each $c \in I$,

$$d(S\psi,S\xi) \le g(d'(\psi,\xi))\{d(\psi(c),S\psi) + d(\xi(c),S\xi) + d'(\psi,\xi)\}.$$

Then, S has a unique PPF dependent fixed point in E_0."

Example 2. Consider

$$S(h) = \frac{5}{9}h\left(\frac{1}{3}\right) + \frac{1}{81}, \text{ for every h} \in E_0$$

$$\psi(x) = \begin{cases} \frac{1}{2} - x^2 \text{ if } x \in [0, \frac{1}{2}] \\ \frac{1}{4} \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$$

$$\xi(x) = \begin{cases} x \text{ if } x \in [0, \frac{1}{3}] \\ \frac{1}{3} \text{ if } x \in [\frac{1}{3}, 1] \end{cases}$$

Clearly, $\psi\left(\frac{1}{3}\right) = \frac{7}{18}$ and $\xi\left(\frac{1}{3}\right) = \frac{1}{3}$, i.e. $\psi\left(\frac{1}{3}\right) \neq \xi\left(\frac{1}{3}\right)$.
Now, define $d(x,y) = |x-y|$, $d'(\psi,\xi) = |\psi - \xi|$ and $g(z_n) = \frac{1}{3} - z_n$, where $z_n = \frac{1}{n}, n \in \mathbb{N}$.
We proceed in a similar manner as we prove in example (1), we find ξ as unique PPF dependent fixed point of S for $c = \frac{19}{81} \in I$.

Theorem 3.7. Let (E, d, α) be a complete SPbMS with parameter α and $S : E_0 \to E$ be a continuous map. Suppose

1. there exists $F \in \mathfrak{F}$ and $\phi \in \Phi$ such that for any $\psi, \xi \in E_0$ with $S\psi \neq S\xi$,
$$F[\alpha d(S\psi, S\xi)] + \phi(d^{'}(\psi,\xi)) \leq F[d^{'}(\psi,\xi)], \tag{3.8}$$

2. for each sequence $\{a_n\} \in \mathbb{R}^+$ such that $\phi(a_n) + F(\alpha a_{n+1}) \leq F(\alpha a_n)$ for each $n \in \mathbb{N}$, then
$$\phi(a_n) + F(\alpha^n a_{n+1}) \leq F(\alpha^{n-1} a_n). \tag{3.9}$$

Then S has exactly one PPF dependent fixed point.

Proof. Define a sequence $\{\psi_n(c)\}$ $\forall$ $n \in \mathbb{N}$, as follow, by using the point ψ_0 in E_0 as an arbitrarily chosen point
$$S\psi_n = \psi_{n+1}(c) = s^{n+1}\psi_0.$$
Assume that there is $p_0 \in \mathbb{N}$ such that $d(\psi_{p_0}(c), S\psi_{p_0+1}(c)) = 0$. Then by (SPbM2) $d(\psi_{p_0}(c), \psi_{p_0}(c)) \leq d(\psi_{p_0}(c), \psi_{p_0+1}(c))$ and $d(\psi_{p_0+1}(c), \psi_{p_0+1}(c)) \leq d(\psi_{p_0}(c), \psi_{p_0+1}(c))$.
So, $d(\psi_{p_0}(c), \psi_{p_0}(c)) = d(\psi_{p_0}(c), \psi_{p_0+1}(c)) = d(\psi_{p_0+1}(c), \psi_{p_0+1}(c))$. Thus, by (SPbM1) $\psi_{p_0}(c) = \psi_{p_0+1}(c)$, that means $\psi_{p_0}(c) = S\psi_{p_0}$, the proof is completed.
So, we assume that $d(\psi_n(c), \psi_{n+1}(c)) > 0$ $\forall$ $n \in \mathbb{N}$.
From inequality (3.8), for all $n \in \mathbb{N}$, we get
$$F(d(S\psi_{n-1}, S\psi_n)) < F(\alpha d(\psi_n(c), \psi_{n+1}(c))) + \phi(d^{'}(\psi_{n-1}, \psi_n) \leq F(d^{'}(\psi_{n-1}, \psi_n),$$

that is

$$F(d(\psi_n(c),\psi_{n+1}(c))) < F(d^{'}(\psi_{n-1},\psi_n)). \tag{3.10}$$

From inequality (3.8) and (3.9), we have

$$F(\alpha^n d(\psi_n(c),\psi_{n+1}(c))) \le F(\alpha^{n-1} d(\psi_{n-1}(c),\psi_n(c))) - \phi(d^{'}(\psi_{n-1},\psi_n)). \tag{3.11}$$

Repeating the same process, we get

$$\begin{aligned} F(\alpha^n d(\psi_n(c),\psi_{n+1}(c))) &\le F(\alpha^{n-1} d(\psi_{n-1}(c),\psi_n(c))) - \phi(d^{'}(\psi_{n-1},\psi_n)) \\ &\le F(\alpha^{n-2} d(\psi_{n-2}(c),\psi_{n-1}(c))) - \phi(d^{'}(\psi_{n-2},\psi_{n-1})) - \phi(d^{'}(\psi_{n-1},\psi_n)) \\ &\le \dots \\ &\le F(d(\psi_0(c),\psi_1(c))) - \sum_{j=0}^{n} \phi(d^{'}(x_j,x_{j+1})). \end{aligned}$$

Since according to our assumption $\liminf_{\alpha\to t^+}\phi(\alpha) > 0$, so

$$\lim\inf_{n\to\infty} \phi(d^{'}(\psi_{n-1},\psi_n)) > 0.$$

Using the definition of limit, $\exists\ n_1 \in \mathbb{N}$ and $c_1 > 0$, so that for each $n \ge n_1$

$$\phi(d^{'}(\psi_{n-1},\psi_n)) \ge c_1.$$

Thus

$$\begin{aligned} F(\alpha^n d(\psi_n(c),\psi_{n+1}(c))) &\le Fd(\psi_0(c),\psi_1(c)) - \sum_{j=0}^{n_1} \phi d^{'}(\psi_j,\psi_{j+1}) - \sum_{j=n_1+1}^{n} \phi(d^{'}(\psi_j,\psi_{j+1})) \\ &\le Fd(\psi_0(c),\psi_1(c)) - \sum_{n_1+1}^{n} c_1 \\ &\le Fd(\psi_0(c),\psi_1(c)) - (n-n_1)c_1. \end{aligned}$$

Applying $\lim n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha^n d(\psi_n(c),\psi_{n+1}(c))) \le \lim_{n\to\infty} [Fd(\psi_0(c),\psi_1(c)) - (n-n_1)c_1]. \tag{3.12}$$

Thus, $\lim_{n\to\infty} F(\alpha^n d(\psi_n(c),\psi_{n+1}(c))) = -\infty$. From condition (F2) of function F, we conclude

$$\lim_{n\to\infty} \alpha^n d(\psi_n(c),\psi_{n+1}(c)) = 0. \tag{3.13}$$

Now, we prove $\lim_{n\to\infty} \alpha^n d(\psi_n(c),\psi_{n+2}(c)) = 0$. Supppose, $\psi_n \ne \psi_p$ for each $n,p \in \mathbb{N}$ with $n \ne p$.

If poosible, let $\psi_n = \psi_p$ for some $n = p + k$, where $k > 0$. Using inequation (3.10), we have

$$d(\psi_p(c),\psi_{p+1}(c)) = d(\psi_n(c),\psi_{n+1}(c)) < d(\psi_{n-1}(c),\psi_n(c)). \tag{3.14}$$

Applying this step again and again, we have

$d(\psi_p(c),\psi_{p+1}(c)) = d(\psi_n(c),\psi_{n+1}(c)) < d(\psi_p(c),\psi_{p+1}(c))$.

From this contradiction, $\psi_n(c) \ne \psi_p(c)$,so $\ \forall\ n,p \in \mathbb{N}\ ,\ \ \psi_n \ne \psi_p$

Now, we prove $d(\psi_n(c),\psi_p(c)) > 0\ \ \forall\ n,p \in \mathbb{N}$, where $n \ne p$. If $d(\psi_n(c),\psi_p(c)) = 0$, by (SPbM2)

$d(\psi_n(c),\psi_n(c)) \leq d(\psi_n(c),\psi_p(c))$ and $d(\psi_p(c),\psi_p(c)) \leq d(\psi_n(c),\psi_p(c))$.
So, $d(\psi_n(c),\psi_n(c)) = d(\psi_p(c),\psi_p(c)) = d(\psi_n(c),\psi_p(c)) = 0$.
Using (SPbM1), $\psi_n(c) = \psi_p(c)$. Again a contradiction. So, $d(\psi_n(c),\psi_p(c)) > 0\, \forall\, n, p \in \mathbb{N}$ and $n \neq p$. Again, using inequality (3.8) and (3.9), we have

$$F(\alpha^n d(\psi_n(c),\psi_{n+2}(c))) \leq F(\alpha^{n-1} d(\psi_{n-1}(c),\psi_{n+1}(c))) - \phi(d^{'}(\psi_{n-1},\psi_{n+1})). \tag{3.15}$$

Repeating the same process, we get,

$$\begin{aligned}
F(\alpha^n d(\psi_n(c),\psi_{n+2}(c))) &\leq F(\alpha^{n-1} d(\psi_{n-1}(c),\psi_{n+1}(c))) - \phi(d^{'}(\psi_{n-1},\psi_{n+1}))\\
&\leq F(\alpha^{n-2} d(\psi_{n-2}(c),\psi_{n}(c))) - \phi(d'(\psi_{n-1},\psi_{n+1})) - \phi(d^{'}(\psi_{n-2},\psi_{n}))\\
&\leq \ldots\\
&\leq F(d(\psi_0(c),\psi_2(c))) - \sum_{j=0}^{n} \phi(d^{'}(\psi_j,\psi_{j+2})).
\end{aligned}$$

According to our assumption $\liminf_{\alpha\to t^+}\phi(\alpha) > 0$, so

$$\lim\inf_{n\to\infty} \phi(d^{'}(\psi_{n-1},\psi_{n+1})) > 0.$$

Using the definition of limit, $\exists\ n_2 \in \mathbb{N}$ and $c_2 > 0$, so that for each $n \geq n_2$

$$\phi(d^{'}(\psi_{n-1},\psi_{n+1})) \geq c_2.$$

Thus

$$\begin{aligned}
F(\alpha^n d(\psi_n(c),\psi_{n+2}(c))) &\leq F d(\psi_0(c),\psi_2(c)) - \sum_{j=0}^{n_2} \phi d^{'}(\psi_j,\psi_{j+2}) - \sum_{j=n_2+1}^{n} \phi(d^{'}(\psi_j,\psi_{j+2}))\\
&\leq F d(\psi_0(c),\psi_2(c)) - \sum_{n_2+1}^{n} c_2\\
&\leq F d(\psi_0(c),\psi_2(c)) - (n-n_2)c_2.
\end{aligned}$$

Applying $\lim n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha^n d(\psi_n(c),\psi_{n+2}(c))) \leq \lim_{n\to\infty}[F d(\psi_0(c),\psi_2(c)) - (n-n_2)c_2]. \tag{3.16}$$

Thus, $\lim_{n\to\infty} F(\alpha^n d(\psi_n(c),\psi_{n+2}(c))) = -\infty$. From condition (F2) of function F, we conclude

$$\lim_{n\to\infty} \alpha^n d(\psi_n(c),\psi_{n+2}(c)) = 0. \tag{3.17}$$

Next, by demonstrating that $\lim_{p,q\to\infty} d(\phi_p(c),\psi_q(c)) = 0$, we demonstrate that $\{\psi_n(c)\}$ is a Cauchy sequence. Using (F2), there exists $k \in (0,1)$, so that

$$\lim_{p\to\infty}[\alpha^p d(\psi_p(c),\psi_{p+1}(c))]^k F(\alpha^p d(\psi_p(c),\psi_{p+1}(c))).$$

Because

$$F[\alpha^p d(\psi_p(c),\psi_{p+1}(c))] \leq F[d(\psi_0(c),\psi_1(c))] - (p-p_1)c_1$$

so,

$$[\alpha^p d(\psi_p(c),\psi_{p+1}(c))]^k F[\alpha^p d(\psi_p(c),\psi_{p+1}(c))] \leq [\alpha^p d(\psi_p(c),\psi_{p+1}(c))]^k [F d(\psi_0(c),\psi_1(c)) - (p-p_1)c_1],$$

that implies

$$[\alpha^p d(\psi_p(c),\psi_{p+1}(c))]^k F[\alpha^p d(\psi_p(c),\psi_{p+1}(c))] \leq [\alpha^p d(\psi_p(c),\psi_{p+1}(c))]^k [F d(\psi_0(c),\psi_1(c))]$$

$$- [(p-p_1)c_1][\alpha^p d(\psi_p(c), \psi_{p+1}(c))]^k.$$

Thus,

$$[\alpha^p d(\psi_p(c), \psi_{p+1}(c))]^k F[\alpha^p d(\psi_p(c), \psi_{p+1}(c))] - \alpha^p d(\psi_p(c), \psi_{p+1}(c))]^k F[d(\psi_0(c), \psi_1(c))]$$
$$\leq -(p-p_1)c_1[\alpha^p d(\psi_p(c), \psi_{p+1}(c))]^k \leq 0.$$

As $n \to \infty$, we conclude

$$\lim_{p\to\infty} (p-p_1)c_1[\alpha^p d(\psi_p(c), \psi_{p+1}(c))]^k = 0.$$

So, $\exists\ h_1 \in \mathbb{N}$, such that for all $p > h_1$

$$\alpha^p d(\psi_p(c), \psi_{p+1}(c)) \leq \frac{1}{[(p-p_1)c_1]^k}. \tag{3.18}$$

Again using (F2), there exists $k \in (0,1)$, so that

$$\lim_{p\to\infty} [\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k F(\alpha^p d(\psi_p(c), \psi_{p+2}(c))).$$

Because

$$F[\alpha^p d(\psi_p(c), \psi_{p+2}(c))] \leq F[d(\psi_0(c), \psi_2(c))] - (p-p_2)c_2,$$

so,

$$[\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k F[\alpha^p d(\psi_p(c), \psi_{p+2}(c))] \leq [\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k [F d(\psi_0(c), \psi_2(c)) - (p-p_2)c_2],$$

that implies

$$[\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k F[\alpha^p d(\psi_p(c), \psi_{p+2}(c))] \leq [\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k [F d(\psi_0(c), \psi_2(c))]$$
$$- [(p-p_2)c_2][\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k.$$

Thus,

$$[\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k F[\alpha^p d(\psi_p(c), \psi_{p+2}(c))] - \alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k F[d(\psi_0(c), \psi_2(c))]$$
$$\leq -(p-p_2)c_2[\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k \leq 0.$$

As $n \to \infty$, we conclude

$$\lim_{p\to\infty} (p-p_2)c_2[\alpha^p d(\psi_p(c), \psi_{p+2}(c))]^k = 0.$$

So, $\exists\ h_2 \in \mathbb{N}$, such that for all $p > h_2$

$$\alpha^p d(\psi_p(c), \psi_{p+2}(c)) \leq \frac{1}{[(p-p_2)c_2]^k}. \tag{3.19}$$

We demonstrate that $\lim_{p\to\infty} d(\psi_p, \psi_{p+q}) = 0$ for each $q \in \mathbb{N}$. The proofs for situations r = 1 and r = 2 are given in equation (3.13) and (3.17). Now taking $q \geq 3$. Examining only two cases will enough.

Case I): Assume $q = 2m+1$, where $m \geq 1$. By using (SPbM4),

$$\begin{aligned} d(\psi_p(c), \psi_{p+q}(c)) &= d(\psi_p(c), \psi_{p+2m+1}(c)) \\ &\leq d(\psi_p(c), \psi_{p+1}(c)) + \alpha(d(\psi_{p+1}(c), \psi_{p+2m+1}(c))) - d(\psi_{p+1}(c), \psi_{p+1}(c)) \\ &\leq d(\psi_p(c), \psi_{p+1}(c)) + \alpha(d(\psi_{p+1}(c), \psi_{p+2m+1}(c))) \\ &\leq d(\psi_p(c), \psi_{p+1}(c)) + \alpha[d(\psi_{p+1}(c), \psi_{p+2}(c)) + \alpha d(\psi_{p+2}(c), \psi_{p+2m+1}(c)) \end{aligned}$$

$$
\begin{aligned}
&\quad - d(\psi_{p+2}(c), \psi_{p+2}(c))] \\
&\leq d(\psi_p(c), \psi_{p+1}(c)) + \alpha d(\psi_{p+1}(c), \psi_{p+2}(c)) + \alpha^2 d(\psi_{p+2}(c), \psi_{p+2m+1}(c)) \leq \ldots \\
&\leq d(\psi_p(c), \psi_{p+1}(c)) + \alpha d(\psi_{p+1}(c), \psi_{p+2}(c)) + \alpha^2 d(\psi_{p+2}(c), \psi_{p+3}(c)) \\
&\quad + \ldots + \alpha^{2m} d(\psi_{p+2m}(c), \psi_{p+2m+1}(c)) \\
&= \frac{1}{\alpha^p} \Big\{ \alpha^p d(\psi_p(c), \psi_{p+1}(c)) + \alpha^{p+1} d(\psi_{p+1}(c), \psi_{p+2}(c)) + \ldots \\
&\quad + \alpha^{p+2m} d(\psi_{p+2m}(c), \psi_{p+2m+1}(c)) \Big\} \\
&= \frac{1}{\alpha^p} \sum_{j=p}^{p+2m} \alpha^j d(\psi_j(c), \psi_{j+1}(c)) \\
&= \frac{1}{\alpha^p} \sum_{j=p}^{p+q-1} \alpha^j d(\psi_j(c), \psi_{j+1}(c)).
\end{aligned}
$$

Thus, for each $p \geq \max\{p_1, p_{h_1}\}$ and $q \in \mathbb{N}$, inequality (3.18) implies

$$
\begin{aligned}
d(\psi_p(c), \psi_{p+q}(c)) &\leq \frac{1}{\alpha^p} \sum_{j=p}^{p+q-1} \alpha^j d(\psi_j(c), \psi_{j+1}(c)) \leq \frac{1}{\alpha^p} \sum_{j=p}^{\infty} \alpha^j d(\psi_j(c), \psi_{j+1}(c)) \\
&\leq \frac{1}{\alpha^p} \sum_{j=p}^{\infty} \frac{1}{[(j-p_1)c_1]^k} \to 0.
\end{aligned}
$$

Case II): Assume $q = 2m$, where $m \geq 1$. By using (SPbM4),

$$
\begin{aligned}
d'(\psi_p, \psi_{p+q}) &= d(\psi_p(c), \psi_{p+2m}(c)) \\
&\leq d(\psi_p(c), \psi_{p+2}(c)) + \alpha(d(\psi_{p+2}(c), \psi_{p+2m}(c))) - d(\psi_{p+2}(c), \psi_{p+2}(c)) \\
&\leq d(\psi_p(c), \psi_{p+2}(c)) + \alpha(d(\psi_{p+2}(c), \psi_{p+2m}(c))) \\
&\leq d(\psi_p(c), \psi_{p+2}(c)) + \alpha[d(\psi_{p+2}(c), \psi_{p+3}(c)) + \alpha d(\psi_{p+3}(c), \psi_{p+2m}(c)) \\
&\quad - d(\psi_{p+3}(c), \psi_{p+3}(c))] \\
&\leq d(\psi_p(c), \psi_{p+2}(c)) + \alpha d(\psi_{p+2}(c), \psi_{p+3}(c)) + \alpha^2 d(\psi_{p+3}(c), \psi_{p+2m}(c)) \leq \ldots \\
&\leq d(\psi_p(c), \psi_{p+2}(c)) + \alpha d(\psi_{p+2}(c), \psi_{p+3}(c)) + \alpha^2 d(\psi_{p+3}(c), \psi_{p+4}(c)) + \ldots \\
&\quad + \alpha^{2m-2} d(\psi_{p+2m-1}(c), \psi_{p+2m}(c)) \\
&= \frac{1}{\alpha^p} \Big\{ \alpha^p d(\psi_p(c), \psi_{p+2}(c)) + \alpha^{p+1} d(\psi_{p+2}(c), \psi_{p+3}(c)) + \ldots \\
&\quad + \alpha^{p+2m-2} d(\psi_{p+2m-1}(c), \psi_{p+2m}(c)) \Big\} \\
&= \frac{1}{\alpha^p} \alpha^p d(\psi_p(c), \psi_{p+2}(c)) + \frac{1}{\alpha^{p+1}} \sum_{j=p+2}^{p+2m-1} \alpha^j d(\psi_j(c), \psi_{j+1}(c))
\end{aligned}
$$

$$= \frac{1}{\alpha^p}\alpha^p d(\psi_p(c), \psi_{p+2}(c)) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{p+q-1}\alpha^j d(\psi_j(c), \psi_{j+1}(c)).$$

Thus, for each $p \geq \max\{p_1, p_2, p_{h_2}\}$ and $q \in \mathbb{N}$, inequality (3.18) and (3.19) implies

$$\begin{aligned}
d(\psi_p(c), \psi_{p+q}(c)) &\leq \frac{1}{\alpha^p}\alpha^p d(\psi_p(c), \psi_{p+2}(c)) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{p+q-1}\alpha^j d(\psi_j(c), \psi_{j+1}(c)) \\
&\leq \frac{1}{\alpha^p}\alpha^p d(\psi_p(c), \psi_{p+2}(c)) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{\infty}\alpha^j d(\psi_j(c), \psi_{j+1}(c)) \\
&\leq \frac{1}{\alpha^p}\left\{\frac{1}{[(p-p_2)c_2]^k} + \frac{1}{\alpha}\sum_{j=p}^{\infty}\frac{1}{[(p-p_1)c_1]^k}\right\} \to 0.
\end{aligned}$$

Thus $\lim_{n\to\infty} d(\psi_n(c), \psi_{p+q}(c)) = 0$.

Hereof, $\{\psi_n\}$ is a Cauchy sequence in E. Because of completeness of (E_0, d), $\exists\ \ \psi^* \in E_0$ such that

$$\lim_{n\to\infty} d^{'}(\psi_n, \psi^*) = 0.$$

We now demonstrate that $d(S\psi^*, \psi^*(c)) = 0$.

By using contradiction as our method of argument $d(S\psi^*, \psi^*(c)) > 0$.

On the other side, F is increing and $F(d(S\psi, S\xi) \leq \phi(d(\psi, \xi)) + F(d(S\psi, S\xi) \leq F(d(\psi(c), \xi(c)))$ for all $\psi, \xi \in E_0$ and $d(S\psi, S\xi) > 0$. We have $d(S\psi, S\xi) \leq d(\psi(c), \xi(c))$ for each $\psi, \xi \in E_0$. This indicates

$$d(S\psi_n, S\psi^*) \leq d(\psi_n(c), \psi^*(c)).$$

As $n \to \infty$, $\ \psi_n \to \psi^*$, then we conclude,

$$\frac{1}{\alpha}d(\psi^*(c), S\psi^*) \leq \lim_{n\to\infty}\sup d(S\psi_n, S\psi^*) \leq \alpha d(\psi^*(c), S\psi^*).$$

So,

$$\frac{1}{\alpha}d(\psi^*(c), S\psi^*) \leq \lim_{n\to\infty}\sup d(S\psi_n, S\psi^*) \leq \lim_{n\to\infty}\sup d'(\psi_n, \psi^*) = 0.$$

Using (SPbM2), $d(S\psi^*, S\psi^*), d(\psi^*(c), \psi^*(c)) \leq d(S\psi^*, \psi^*(c))$. Thus $S\psi^* = \psi^*(c)$.

To demonstrate uniqueness, assume $\psi^*, \xi^* \in E_0$ are different PPF dependent fixed points of E_0. So,

$$d^{'}(\psi^*, \xi^*) = d(S\psi^*, S\xi^*) > 0.$$

Using inequation (3.8), we get

$$\begin{aligned}
F(d(\psi^*(c), \xi^*(c))) &= F(d(S\psi^*, S\xi^*)) \\
&\leq F(\alpha d(S\psi^*, S\xi^*)) \\
&\leq F(d(\psi^*(c), \xi^*(c))) - \phi(d(\psi^*(c), \xi^*(c))) \\
&< F(d(\psi^*(c), \xi^*(c)))
\end{aligned}$$

Here, we have a contradiction. Hence $\psi^*(c) = \xi^*(c) \quad \forall \quad c \in [a,b]$. So, $\psi^* = \xi^*$. This completes the proof. $\square$

Corollary 3.8. If we replace codition (i) of Theorem (3.7) by

$$\alpha d(S\psi, S\xi) \le e^{\frac{-1}{d'(\psi,\xi)+1}},$$

for each $\psi, \xi \in E_0$ such that $S\psi \ne S\xi$. Then S has only one PPF dependent fixed point.

Proof. By applying logrithm on both sides, we get

$$\begin{aligned}\log(\alpha d(S\psi, S\xi)) &\le \log\left[\frac{-1}{d'(u,\xi)+1}\right] \\ &= \log(d^{'}(\psi,\xi)) + \frac{-1}{d'(\psi,\xi)+1}\end{aligned}$$

With $\phi(z) = \frac{1}{z+1}$ and $F(z) = \log(z)$, we find the same inequality (3.8). Hence the proof. $\square$

Example 3. Consider the function S, ψ, ξ, d, d', f defined in example (1). Define $F(t) = ln(t) + t$ with $\phi(s) = \frac{1}{2} + s$ and $\alpha = \frac{26}{25}$. We have

$$d(S\psi, S\xi) = \frac{5}{162}, (d^{'}(\psi,\xi) = \frac{1}{18}$$

From condition (3.8), we calculated

$$\begin{aligned}F[\alpha d(S\psi, S\xi)] + \phi(d^{'}(\psi,\xi)) &= F\left[\frac{26}{25}\frac{5}{162}\right] + \left(\frac{1}{2} + \frac{1}{18}\right) \\ &= \frac{10}{18} + F[0.032] \\ &= \frac{10}{18} + ln(0.032) + 0.032 \\ &= -2.854\end{aligned}$$

and

$$\begin{aligned}F[d^{'}(\psi,\xi)] &= ln\left(\frac{1}{18}\right) + \frac{1}{18} \\ &= -2.834\end{aligned}$$

Thus, the conditions of Theorem (3.7) are satisfied and ξ is PPF dependent fixed point of S for $c = \frac{10}{81} \in I$.

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