TAMKANG JOURNAL OF MATHEMATICS
Volume 55, Number 3, 251-265, 2024
DOI:10.5556/j.tkjm.55.2024.5145

Some fixed point results for nonlinear F-type contractions in strong partial b-metric spaces

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Abstract. In this article, we demonstrate some fixed point results that generalises the Banach contraction principle in a different way from the previously established literature findings. We provide some fixed point findings for nonlinear F type contractions in Strong Partial b-Metric Spaces (SPbMS). We also include some examples that demonstrates the applicability of our findings.

Keywords. Fixed point, uniqueness, SPbMS

1 Introduction

Fixed point theory has emerged as a highly useful tool in the study of nonlinear processes during the last few decades. Fixed point concepts and findings in pure and applied analysis, topology, and geometry have been developed in particular. The well-known Banach contraction principle [4] is a key of this theory. It has been widely investigated and extended in a variety of scenarios ([6],[7],[12],[13],[19],[20],[18],[22]). The works of Bourbaki [5] and Bakhtin [3] influenced the concept of b-metric. In 1993, Czerwik [8] provided a weaker assumption than the triangle inequality and explicitly defined a b-metric space in order to generalise the Banach contraction mapping theorem. Matthews [16], in 1994 proposed the concept of partial metric space as part of the research of denotational semantics of dataflow networks and demonstrated how the Banach contraction principle may be adapted to the partial metric context for programme verification applications. In [17], the notion of SPbMSs was introduced. They also discussed the relationship between strong b-metric and SPbMSs.

A novel concept of contraction known as F-contraction was first proposed by Wardowski [23]. As a result, Wardowski demonstrated fixed point theorems in a novel manner that differed from how the prior known theorems of the same class had been established, generalising the Banach-Caccioppoli fixed point theorem. In 2014, Jleli et al. [10],[9] examined an extension of the Banach fixed point theorem in a brand-new field of contraction mappings on metric spaces known as θ contraction. In a new class of contraction mappings on metric spaces known as (θ, F)-contraction (*nonlinear F-contraction*), Wardowski [24] examined an extension of the Banach fixed point theorem in 2018.

Inspired by the outcomes of Kari et al. [11] and Wardowski [23], we establish some fixed point results for nonlinear F-contraction type mappings in the case of SPbMSs.

Received date: June 4, 2023; Published online: September 1, 2023.
2010 *Mathematics Subject Classification*. 47H10, 54H25, 54E50.
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2 Preliminaries

Here, we provide the relevant definitions and findings for different spaces and different type of contractions that will be helpful for further explanation.

Definition 1. [16] "A partial metric on a set E is a function $d : E \times E \to \mathbb{R}_0^+$ such that for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$, the following conditions hold:

(PM1) $\mathfrak{a} = \mathfrak{b} \Leftrightarrow d(\mathfrak{a}, \mathfrak{a}) = d(\mathfrak{b}, \mathfrak{b}) = d(\mathfrak{a}, \mathfrak{b})$;

(PM2) $d(\mathfrak{a}, \mathfrak{a}) \leq d(\mathfrak{a}, \mathfrak{b})$;

(PM3) $d(\mathfrak{a}, \mathfrak{b}) = d(\mathfrak{b}, \mathfrak{a})$;

(PM4) $d(\mathfrak{a}, \mathfrak{b}) \leq d(\mathfrak{a}, \mathfrak{c}) + d(\mathfrak{c}, \mathfrak{b}) - d(\mathfrak{c}, \mathfrak{c})$.

Then (E, d) is called a partial metric space."

Definition 2. [14] "A map $d : E \times E \to \mathbb{R}_0^+$ is a strong b-metric on a non empty set E if for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$ and $\alpha \geq 1$ the following conditions hold:

(SB1) $\mathfrak{a} = \mathfrak{b}$ iff $d(\mathfrak{a}, \mathfrak{b}) = 0$;

(SB2) $d(\mathfrak{a}, \mathfrak{b}) = d(\mathfrak{b}, \mathfrak{a})$;

(SB3) $d(\mathfrak{a}, \mathfrak{b}) \leq d(\mathfrak{a}, \mathfrak{c}) + \alpha d(\mathfrak{c}, \mathfrak{b})$.

The triple (E, d, α) is called a strong b-metric space."

Definition 3. [17] "A map $d : E \times E \to \mathbb{R}_0^+$ is a strong partial b-metric on a non empty set E if for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$ and $\alpha \geq 1$ the following conditions hold:

(SPbM1) $\mathfrak{a} = \mathfrak{b} \Leftrightarrow d(\mathfrak{a}, \mathfrak{a}) = d(\mathfrak{b}, \mathfrak{b}) = d(\mathfrak{a}, \mathfrak{b})$;

(SPbM2) $d(\mathfrak{a}, \mathfrak{a}) \leq d(\mathfrak{a}, \mathfrak{b})$;

(SPbM3) $d(\mathfrak{a}, \mathfrak{b}) = d(\mathfrak{b}, \mathfrak{a})$;

(SPbM4) $d(\mathfrak{a}, \mathfrak{b}) \leq d(\mathfrak{a}, \mathfrak{c}) + \alpha d(\mathfrak{c}, \mathfrak{b}) - d(\mathfrak{c}, \mathfrak{c})$.

The triple (E, d, α) is called a Strong Partial b-Metric Space (SPbM)."

Remark 1. [17] "Every metric space is a strong b-metric space but converse is not neccessarily true. Every strong b-metric space is a SPbM but not conversely."

Definition 4. [17] " Let (E, d, α) be a SPbM. Then

(i) A sequence $\{\mathfrak{a}_n\}$ in (E, d, α) converges to a point $\mathfrak{a} \in E$ if $d(\mathfrak{a}, \mathfrak{a}) = \lim_n d(\mathfrak{a}_n, \mathfrak{a}) = \lim_n d(\mathfrak{a}_n, \mathfrak{a}_n)$.

(ii) A sequence $\{\mathfrak{a}_n\}$ in (E, d, α) is Cauchy if the $\lim_{n,m} d(\mathfrak{a}_n, \mathfrak{a}_m)$ exists and finite."

Definition 5. [23] "Let $\mathfrak{F}$ be the family of all continuous functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

(F1) F is strictly increasing;

(F2) For each sequence $\{a_n\} \in \mathbb{N}$ of positive numbers

$$\lim_{n\to\infty} a_n = 0 \text{ if and only if } \lim_{n\to\infty} F(a_n) = -\infty; \tag{2.1}$$

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $S : E \times E$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $u, v \in E$

$$(d(Su, Sv)) > 0 \implies \tau + F(d(Su, Sv)) \le F(d(u, v)).\text{''} \tag{2.2}$$

Turinici [21] observed that the condition (F2) can be relaxed to the form
(F2') $\lim_{n \to \infty} F(a_n) = -\infty$.

Definition 6. [24] "A mapping $S : E \to E$ is said to be a (ϕ, F)-contraction (or nonlinear F-contraction) if there exist the functions $F : (0, \infty) \to \mathbb{R}$ and $\phi : (0, \infty) \to (0, \infty)$ satisfying the following

(H1) F satisfies (F1) and (F2');

(H2) $\liminf_{s \to t^+} \phi(s) > 0$ for all $t \ge 0$;

(H3) $\phi(d(u, v)) + F(d(Su, Sv)) \le F(d(u, v))$ for all $u, v \in E$ such that $Su \ne Sv$."

Theorem 2.1. *[24]* "Let (E, d) be a complete metric space and $S : E \to E$ be a (ϕ, F)-contraction. Then S has a unique fixed point."

3 Main Results

Throughout the paper, $\mathfrak{F}$ is a family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfies (F1), (F2), (F2'), (F3). $\mathbb{R}$ represents the set of real numbers and $\mathbb{N}$ is the set of natural numbers. Φ is the family of all functions $\phi : (0, \infty) \to (0, \infty)$ with the condition $\liminf_{s \to t^+} \phi(s) > 0$ for all $t \ge 0$.

Theorem 3.1. Let (E, d, α) be a complete SPbMS with parameter α and $S : E \to E$ be a continuous map. Suppose

(i) there exist $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u, v \in E$ with $Su \ne Sv$,

$$F[\alpha d(Su, Sv)] + \phi(d(u, v)) \le F[d(u, v)], \tag{3.1}$$

(ii) for each sequence $\{a_n\} \in \mathbb{R}^+$ such that $\phi(a_n) + F(\alpha a_{n+1}) \le F(\alpha a_n)$ for each $n \in \mathbb{N}$, then

$$\phi(a_n) + F(\alpha^n a_{n+1}) \le F(\alpha^{n-1} a_n). \tag{3.2}$$

Then S has exactly one fixed point.

Proof. Define a sequence $\{u_n\}$ $\forall$ $n \in \mathbb{N}$, as follow, by using the point u_0 in E as an arbitrarily chosen point

$$Su_n = u_{n+1} = s^{n+1} u_0.$$

Assume that there is $p_0 \in \mathbb{N}$ such that $d(u_{p_0}, u_{p_0+1}) = 0$. Then by (SPbM2)
$d(u_{p_0}, u_{p_0}) \le d(u_{p_0}, u_{p_0+1})$ and $d(u_{p_0+1}, u_{p_0+1}) \le d(u_{p_0}, u_{p_0+1})$.
So, $d(u_{p_0}, u_{p_0}) = d(u_{p_0}, u_{p_0+1}) = d(u_{p_0+1}, u_{p_0+1})$. Thus, by (SPbM1) $u_{p_0} = u_{p_0+1}$, the proof is completed.
So, we assume that $d(u_n, u_{n+1}) > 0$ $\forall$ $n \in \mathbb{N}$.
From inequality (3.1), for all $n \in \mathbb{N}$, we get

$$F(d(Su_{n-1}, Su_n)) < F(\alpha d(u_n, u_{n+1})) + \phi(d(u_{n-1}, u_n) \le F(d(u_{n-1}, u_n),$$

that is

$$F(d(u_n, u_{n+1})) < F(d(u_{n-1}, u_n)). \tag{3.3}$$

From inequality (3.1) and (3.2), we have

$$F(\alpha^n d(u_n,u_{n+1})) \leq F(\alpha^{n-1} d(u_{n-1},u_n)) - \phi(d(u_{n-1},u_n)). \tag{3.4}$$

Repeating the same process, we get

$$\begin{aligned} F(\alpha^n d(u_n,u_{n+1})) &\leq F(\alpha^{n-1} d(u_{n-1},u_n)) - \phi(d(u_{n-1},u_n)) \\ &\leq F(\alpha^{n-2} d(u_{n-2},u_{n-1})) - \phi(d(u_{n-2},u_{n-1})) - \phi(d(u_{n-1},u_n)) \\ &\leq \dots \\ &\leq F(d(u_0,u_1)) - \sum_{j=0}^{n} \phi(d(x_j,x_{j+1})). \end{aligned}$$

Since according to our assumption $\liminf_{\alpha\to t^+} \phi(\alpha) > 0$, so

$$\lim_{n\to\infty}\inf \phi(d(u_{n-1},u_n)) > 0.$$

Using the definition of limit, $\exists\ n_1 \in \mathbb{N}$ and $c_1 > 0$, so that for each $n \geq n_1$

$$\phi(d(u_{n-1},u_n)) \geq c_1.$$

Thus

$$\begin{aligned} F(\alpha^n d(u_n,u_{n+1})) &\leq Fd(u_0,u_1) - \sum_{j=0}^{n_1} \phi d(u_j,u_{j+1}) - \sum_{j=n_1+1}^{n} \phi(d(u_j,u_{j+1})) \\ &\leq Fd(u_0,u_1) - \sum_{n_1+1}^{n} c_1 \\ &\leq Fd(u_0,u_1) - (n-n_1)c_1. \end{aligned}$$

Applying $\lim n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha^n d(u_n,u_{n+1})) \leq \lim_{n\to\infty} [Fd(u_0,u_1) - (n-n_1)c_1]. \tag{3.5}$$

Thus, $\lim_{n\to\infty} F(\alpha^n d(u_n,u_{n+1})) = -\infty$. From condition (F2) of function F, we conclude

$$\lim_{n\to\infty} \alpha^n d(u_n,u_{n+1}) = 0. \tag{3.6}$$

Now, we prove $\lim_{n\to\infty} \alpha^n d(u_n,u_{n+2}) = 0$. Supppose, $u_n \neq u_p$ for each $n,p \in \mathbb{N}$ with $n \neq p$. If possible, let $u_n = u_p$ for some $n = p+k$, where $k > 0$. Using inequation (3.3), we have

$$d(u_p,u_{p+1}) = d(u_n,u_{n+1}) < d(u_{n-1},u_n). \tag{3.7}$$

Applying this step again and again, we have $d(u_p,u_{p+1}) = d(u_n,u_{n+1}) < d(u_p,u_{p+1})$.
From this contradiction, $u_n \neq u_p\ \ \forall\ n,p \in \mathbb{N}$.
Now, we prove $d(u_n,u_p) > 0\ \ \forall\ n,p \in \mathbb{N}$, where $n \neq p$. If $d(u_n,u_p) = 0$, by (SPbM2) $d(u_n,u_n) \leq d(u_n,u_p)$ and $d(u_p,u_p) \leq d(u_n,u_p)$.
So, $d(u_n,u_n) = d(u_p,u_p) = d(u_n,u_p) = 0$.
Using (SPbM1), $u_n = u_p$. Again a contradiction. So, $d(u_n,u_p) > 0\ \forall\ n,p \in \mathbb{N}$ and $n \neq p$. Again, using inequality (3.1) and (3.2), we have

$$F(\alpha^n d(u_n,u_{n+2})) \leq F(\alpha^{n-1} d(u_{n-1},u_{n+1})) - \phi(d(u_{n-1},u_{n+1})). \tag{3.8}$$

Repeating the same process, we get,

$$\begin{aligned} F(\alpha^n d(u_n,u_{n+2})) &\leq F(\alpha^{n-1} d(u_{n-1},u_{n+1})) - \phi(d(u_{n-1},u_{n+1})) \\ &\leq F(\alpha^{n-2} d(u_{n-2},u_n)) - \phi(d(u_{n-1},u_{n+1})) - \phi(d(u_{n-2},u_n)) \\ &\leq \dots \\ &\leq F(d(u_0,u_2)) - \sum_{j=0}^{n} \phi(d(x_j,x_{j+2})). \end{aligned}$$

According to our assumption $\liminf_{\alpha\to t^+} \phi(\alpha) > 0$, so

$$\lim_{n\to\infty}\inf \phi(d(u_{n-1},u_{n+1})) > 0.$$

Using the definition of limit, $\exists\ n_2 \in \mathbb{N}$ and $c_2 > 0$, so that for each $n \geq n_2$

$$\phi(d(u_{n-1}, u_{n+1})) \geq c_2.$$

Thus

$$\begin{aligned} F(\alpha^n d(u_n, u_{n+2})) &\leq Fd(u_0, u_2) - \sum_{j=0}^{n_2} \phi d(u_j, u_{j+2}) - \sum_{j=n_2+1}^{n} \phi(d(u_j, u_{j+2})) \\ &\leq Fd(u_0, u_2) - \sum_{n_2+1}^{n} c_2 \\ &\leq Fd(u_0, u_2) - (n - n_2)c_2. \end{aligned}$$

Applying $\lim n \to \infty$, we have

$$\lim_{n\to\infty} F(\alpha^n d(u_n, u_{n+2})) \leq \lim_{n\to\infty} [Fd(u_0, u_2) - (n - n_2)c_2]. \tag{3.9}$$

Thus, $\lim_{n\to\infty} F(\alpha^n d(u_n, u_{n+2})) = -\infty$. From condition (F2) of function F, we conclude

$$\lim_{n\to\infty} \alpha^n d(u_n, u_{n+2}) = 0. \tag{3.10}$$

Next, by demonstrating that $\lim_{p,q\to\infty} d(u_p, u_q) = 0$, we demonstrate that $\{u_n\}$ is a Cauchy sequence. Using (F2), there exists $k \in (0, 1)$, so that

$$\lim_{p\to\infty} [\alpha^p d(u_p, u_{p+1})]^k F(\alpha^p d(u_p, u_{p+1})).$$

Because

$$F[\alpha^p d(u_p, u_{p+1})] \leq F[d(u_0, u_1)] - (p - p_1)c_1$$

so,

$$[\alpha^p d(u_p, u_{p+1})]^k F[\alpha^p d(u_p, u_{p+1})] \leq [\alpha^p d(u_p, u_{p+1})]^k [Fd(u_0, u_1) - (p - p_1)c_1],$$

that implies

$$[\alpha^p d(u_p, u_{p+1})]^k F[\alpha^p d(u_p, u_{p+1})] \leq [\alpha^p d(u_p, u_{p+1})]^k [Fd(u_0, u_1)] - [(p - p_1)c_1][\alpha^p d(u_p, u_{p+1})]^k.$$

Thus,

$$[\alpha^p d(u_p, u_{p+1})]^k F[\alpha^p d(u_p, u_{p+1})] - \alpha^p d(u_p, u_{p+1})]^k F[d(u_0, u_1)] \leq -(p - p_1)c_1[\alpha^p d(u_p, u_{p+1})]^k \leq 0.$$

As $n \to \infty$, we conclude

$$\lim_{p\to\infty} (p - p_1)c_1[\alpha^p d(u_p, u_{p+1})]^k = 0.$$

So, $\exists\ h_1 \in \mathbb{N}$, such that for all $p > h_1$

$$\alpha^p d(u_p, u_{p+1}) \leq \frac{1}{[(p - p_1)c_1]^k}. \tag{3.11}$$

Again using (F2), there exists $k \in (0, 1)$, so that

$$\lim_{p\to\infty} [\alpha^p d(u_p, u_{p+2})]^k F(\alpha^p d(u_p, u_{p+2})).$$

Because

$$F[\alpha^p d(u_p, u_{p+2})] \leq F[d(u_0, u_2)] - (p - p_2)c_2.$$

So,

$$[\alpha^p d(u_p, u_{p+2})]^k F[\alpha^p d(u_p, u_{p+2})] \leq [\alpha^p d(u_p, u_{p+2})]^k [Fd(u_0, u_2) - (p - p_2)c_2],$$

that implies

$$[\alpha^p d(u_p, u_{p+2})]^k F[\alpha^p d(u_p, u_{p+2})] \leq [\alpha^p d(u_p, u_{p+2})]^k [Fd(u_0, u_2)] - [(p - p_2)c_2][\alpha^p d(u_p, u_{p+2})]^k.$$

Thus,

$$[\alpha^p d(u_p, u_{p+2})]^k F[\alpha^p d(u_p, u_{p+2})] - \alpha^p d(u_p, u_{p+2})]^k F[d(u_0, u_2)] \leq -(p - p_2)c_2[\alpha^p d(u_p, u_{p+2})]^k \leq 0.$$

As $n \to \infty$, we conclude

$$\lim_{p\to\infty} (p - p_2)c_2[\alpha^p d(u_p, u_{p+2})]^k = 0.$$

So, $\exists\ h_2 \in \mathbb{N}$, such that for all $p > h_2$

$$\alpha^p d(u_p, u_{p+2}) \leq \frac{1}{[(p-p_2)c_2]^k}. \tag{3.12}$$

We demonstrate that $\lim p \to \infty d(u_p, u_{p+q}) = 0$ for each $q \in \mathbb{N}$.
The proofs for situations r = 1 and r = 2 are given in equation (3.6) and (3.10).
Now taking $q \geq 3$. Examining only two cases are enough.
Case I): Assume $q = 2m+1$, where $m \geq 1$. By using (SPbM4),

$$\begin{aligned} d(u_p, u_{p+q}) &= d(u_p, u_{p+2m+1}) \\ &\leq d(u_p, u_{p+1}) + \alpha(d(u_{p+1}, u_{p+2m+1})) - d(u_{p+1}, u_{p+1}) \\ &\leq d(u_p, u_{p+1}) + \alpha(d(u_{p+1}, u_{p+2m+1})) \\ &\leq d(u_p, u_{p+1}) + \alpha[d(u_{p+1}, u_{p+2}) + \alpha d(u_{p+2}, u_{p+2m+1}) - d(u_{p+2}, u_{p+2})] \\ &\leq d(u_p, u_{p+1}) + \alpha d(u_{p+1}, u_{p+2}) + \alpha^2 d(u_{p+2}, u_{p+2m+1}) \leq ... \\ &\leq d(u_p, u_{p+1}) + \alpha d(u_{p+1}, u_{p+2}) + \alpha^2 d(u_{p+2}, u_{p+3}) + ... + \alpha^{2m} d(u_{p+2m}, u_{p+2m+1}) \\ &= \frac{1}{\alpha^p}\left\{\alpha^p d(u_p, u_{p+1}) + \alpha^{p+1} d(u_{p+1}, u_{p+2}) + ... + \alpha^{p+2m} d(u_{p+2m}, u_{p+2m+1})\right\} \\ &= \frac{1}{\alpha^p}\sum_{j=p}^{p+2m} \alpha^j d(u_j, u_{j+1}) \\ &= \frac{1}{\alpha^p}\sum_{j=p}^{p+q-1} \alpha^j d(u_j, u_{j+1}). \end{aligned}$$

Thus, for each $p \geq \max\{p_1, p_{h_1}\}$ and $q \in \mathbb{N}$, inequality (3.11) implies

$$d(u_p, u_{p+q}) \leq \frac{1}{\alpha^p}\sum_{j=p}^{p+q-1} \alpha^j d(u_j, u_{j+1}) \leq \frac{1}{\alpha^p}\sum_{j=p}^{\infty} \alpha^j d(u_j, u_{j+1}) \leq \frac{1}{\alpha^p}\sum_{j=p}^{\infty} \frac{1}{[(j-p_1)c_1]^k} \to 0.$$

Case II): Assume $q = 2m$, where $m \geq 1$. By using (SPbM4),

$$\begin{aligned} d(u_p, u_{p+q}) &= d(u_p, u_{p+2m}) \\ &\leq d(u_p, u_{p+2}) + \alpha(d(u_{p+2}, u_{p+2m})) - d(u_{p+2}, u_{p+2}) \\ &\leq d(u_p, u_{p+2}) + \alpha(d(u_{p+2}, u_{p+2m})) \\ &\leq d(u_p, u_{p+2}) + \alpha[d(u_{p+2}, u_{p+3}) + \alpha d(u_{p+3}, u_{p+2m}) - d(u_{p+3}, u_{p+3})] \\ &\leq d(u_p, u_{p+2}) + \alpha d(u_{p+2}, u_{p+3}) + \alpha^2 d(u_{p+3}, u_{p+2m}) \leq ... \\ &\leq d(u_p, u_{p+2}) + \alpha d(u_{p+2}, u_{p+3}) + \alpha^2 d(u_{p+3}, u_{p+4}) + ... + \alpha^{2m-2} d(u_{p+2m-1}, u_{p+2m}) \\ &= \frac{1}{\alpha^p}\left\{\alpha^p d(u_p, u_{p+2}) + \alpha^{p+1} d(u_{p+2}, u_{p+3}) + ... + \alpha^{p+2m-2} d(u_{p+2m-1}, u_{p+2m})\right\} \\ &= \frac{1}{\alpha^p}\alpha^p d(u_p, u_{p+2}) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{p+2m-1} \alpha^j d(u_j, u_{j+1}) \\ &= \frac{1}{\alpha^p}\alpha^p d(u_p, u_{p+2}) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{p+q-1} \alpha^j d(u_j, u_{j+1}). \end{aligned}$$

Thus, for each $p \geq \max\{p_1, p_2, p_{h_2}\}$ and $q \in \mathbb{N}$, inequality (3.11) and (3.12) implies

$$d(u_p, u_{p+q}) \leq \frac{1}{\alpha^p}\alpha^p d(u_p, u_{p+2}) + \frac{1}{\alpha^{p+1}}\sum_{j=p+2}^{p+q-1} \alpha^j d(u_j, u_{j+1})$$

$$\leq \frac{1}{\alpha^p}\alpha^p d(u_p, u_{p+2}) + \frac{1}{\alpha^{p+1}} \sum_{j=p+2}^{\infty} \alpha^j d(u_j, u_{j+1})$$
$$\leq \frac{1}{\alpha^p}\left\{ \frac{1}{[(p-p_2)c_2]^k} + \frac{1}{\alpha}\sum_{j=p}^{\infty} \frac{1}{[(p-p_1)c_1]^k} \right\} \to 0.$$

Thus $\lim_{n\to\infty} d(u_n, u_{p+q}) = 0$.
Hereof, $\{u_n\}$ is a Cauchy sequence in E. Because of completeness of (E, d), $\exists\ u^* \in E$ such that
$$\lim_{n\to\infty} d(u_n, u^*) = 0.$$
We now demonstrate that $d(Su^*, u^*) = 0$.
By using contradiction to our method of argument $d(Su^*, u^*) > 0$.
On the other side, F is incresing and
$F(d(Su, Sv) \leq \phi(d(u, v)) + F(d(Su, Sv) \leq F(d(u, v))$ for all $u, v \in E$ and $d(Su, Sv) > 0$. We have $d(Su, Sv) \leq d(u, v)$ for each $u, v \in E$. This indicates
$$d(Su_n, Su^*) \leq d(u_n, u^*).$$
As $n \to \infty$, $u_n \to u^*$, then we conclude,
$$\frac{1}{\alpha} d(u^*, Su^*) \leq \lim_{n\to\infty} \sup d(Su_n, Su^*) \leq \alpha d(u^*, Su^*).$$
So,
$$\frac{1}{\alpha} d(u^*, Su^*) \leq \lim_{n\to\infty} \sup d(Su_n, Su^*) \leq \lim_{n\to\infty} \sup d(u_n, u^*) = 0.$$
Using (SPbM2), $d(Su^*, Su^*), d(u^*, u^*) \leq d(Su^*, u^*)$. Thus $Su^* = u^*$. To demonstrate uniqueness, assume $u^*, v^* \in E$ are different fixed points of E. So,
$$d(u^*, v^*) = d(Su^*, Sv^*) > 0.$$
Using inequation (3.1), we get
$$\begin{aligned} F(d(u^*, v^*)) &= F(d(Su^*, Sv^*)) \\ &\leq F(\alpha d(Su^*, Sv^*)) \\ &\leq F(d(u^*, v^*)) - \phi(d(u^*, v^*)) \\ &< F(d(u^*, v^*)) \end{aligned}$$

Here, we have a contradiction. Hence $u^* = v^*$. This completes the proof.

Corollary 3.2. If we replace codition (i) of Theorem (3.1) by
$$\alpha d(Su, Sv) \leq e^{\frac{-1}{d(u,v)+1}},$$
for each $u, v \in E$ such that $Su \neq Sv$. Then S has only one fixed point.

Proof. By applying logrithm on both sides, we get
$$\begin{aligned} \log(\alpha d(Su, Sv)) &\leq \log\left[\frac{-1}{d(u, v) + 1}\right] \\ &= \log(d(u, v)) + \frac{-1}{d(u, v) + 1} \end{aligned}$$
With $\phi(z) = \frac{1}{z+1}$ and $F(z) = \log(z)$, we find the same inequality (3.1). Hence the proof.

Example 1. Let $E = \{0, 1, 2\}$ and $d : E \times E \to [0, \infty)$ be defined by
$$d(0, 0) = d(2, 2) = 0, d(1, 1) = \frac{1}{4},$$
$$d(1, 0) = \frac{1}{2} = d(0, 1),$$

$$d(1,2) = 6 = d(2,1),$$
$$d(2,0) = 8 = d(0,2).$$

Here $d(u,u) \leq d(u,v) \ \ \forall \ \ u,v \in E$. And

$$\begin{aligned}
d(0,1) &\leq d(0,2) + \alpha d(2,1) - d(2,2), \quad \forall \ \ \alpha \geq 1,\\
d(1,0) &\leq d(1,2) + \alpha d(2,0) - d(2,2), \quad \forall \ \ \alpha \geq 1,\\
d(0,2) &\leq d(0,1) + \alpha d(1,2) - d(1,1), \quad \forall \ \ \alpha \geq \frac{31}{24},\\
d(2,0) &\leq d(2,1) + \alpha d(1,0) - d(1,1), \quad \forall \ \ \alpha \geq \frac{9}{2},\\
d(1,2) &\leq d(1,0) + \alpha d(0,2) - d(0,0), \quad \forall \ \ \alpha \geq 1,\\
d(2,1) &\leq d(2,0) + \alpha d(0,1) - d(2,2), \quad \forall \ \ \alpha \geq 1.
\end{aligned}$$

So, (E,d,α) is a SPbMS, for $\alpha = 5$ but it is neither metric nor strong b-metric space, because $d(1,1) = \frac{1}{4} \neq 0$.

Let $S : E \to E$ be a self map defined by $S0 = 0, S1 = 0, S2 = 1$ and $f \in \mathfrak{F}$ and $\phi \in \Phi$ be represented as

$$F(u) = \log(u), \qquad \phi(u) = \frac{1}{u+1}.$$

For $Su \neq Sv$, we have only two choices $(u,v) = (0,2)$ and $(u,v) = (1,2)$.

If $(u,v) = (0,2)$, then

$$\begin{aligned}
F[\alpha d(Su,Sv)] + \phi(d(u,y)) - F[d(u,y)] &= F[5(d(S0,S2))] + \phi d(0,2) - F[d(0,2)]\\
&= F[5(d(0,1))] + \phi d(0,2) - F[d(0,2)]\\
&= F[5(\frac{1}{2})] + \phi(8) - F(8)\\
&= \log\frac{5}{2} + \frac{1}{9} - \log(8)\\
&= 0.3979 + 0.1111 - 0.9030 = -0.394 < 0.
\end{aligned}$$

If $(u,v) = (1,2)$, then

$$\begin{aligned}
F[\alpha d(Su,Sv)] + \phi(d(u,y)) - F[d(u,y)] &= F[5(d(S1,S2))] + \phi d(1,2) - F[d(1,2)]\\
&= F[5(d(0,1))] + \phi d(1,2) - F[d(1,2)]\\
&= F[5(\frac{1}{2})] + \phi(6) - F(6)\\
&= \log\frac{5}{2} + \frac{1}{7} - \log(6)\\
&= 0.3979 + 0.1428 - 0.7781 = -0.2374 < 0.
\end{aligned}$$

As a result, all of the requirements are met. Therefore, S has a single fixed point $u^* = 0$.

Corollary 3.3. "Let (E,d,α) be a complete strong b-metric space with parameter α and $S : E \to E$ be a continuous map. Suppose

(i) there exists $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u,v \in E$ with $Su \neq Sv$,
$$F[\alpha d(Su,Sv)] + \phi(d(u,v)) \leq F[d(u,v)],$$

(ii) for each sequence $\{a_n\} \in \mathbb{R}^+$ such that $\phi(a_n) + F(\alpha a_{n+1}) \leq F(\alpha a_n)$ for each $n \in \mathbb{N}$, then
$$\phi(a_n) + F(\alpha^n a_{n+1}) \leq F(\alpha^{n-1} a_n).$$

Then S has exactly one fixed point."

Corollary 3.4. "Let (E, d, α) be a complete metric space with parameter α and $S : E \to E$ be a continuous map. Suppose

(i) there exists $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u, v \in E$ with $Su \neq Sv$,
$$F[\alpha d(Su, Sv)] + \phi(d(u, v)) \leq F[d(u, v)],$$

(ii) for each sequence $\{a_n\} \in \mathbb{R}^+$ such that $\phi(a_n) + F(\alpha a_{n+1}) \leq F(\alpha a_n)$ for each $n \in \mathbb{N}$, then
$$\phi(a_n) + F(\alpha^n a_{n+1}) \leq F(\alpha^{n-1} a_n).$$

Then S has exactly one fixed point."

Lemma 3.1. Let (E, d) be a SPbMS and $\{u_n\}$ be a sequence in E, so as
$$\lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(u_n, u_{n+2}) = 0. \tag{3.13}$$
If $\{u_n\}$ is not a Cauchy sequence, then there must be a positive δ and two sequence of positive numbers, $\{p_k\}$ and $\{q_k\}$ such that
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_k}, u_{q_k}) \leq \lim_{k\to\infty} \sup d(u_{p_k}, u_{q_k}) \leq \alpha\delta,$$
$$\delta \leq \lim_{k\to\infty} \inf d(u_{q_k}, u_{p_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{q_k}, u_{p_{k+1}}) \leq \alpha\delta,$$
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_k}, u_{q_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{p_k}, u_{q_{k+1}}) \leq \alpha\delta,$$
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_{k+1}}, u_{q_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{p_{k+1}}, u_{q_{k+1}}) \leq \alpha^2\delta.$$

Proof. According to the definition of Cauchy sequence, if $\{u_n\}$ is not Cauchy, then there exist a $\delta > 0$, and two sequence of positive numbers $\{p_k\}$ and $\{q_k\}$, such as for $p_k > q_k > k$,
$$\delta \leq d(u_{p_k}, u_{q_k}) \quad \text{and} \quad d(u_{p_{k-1}}, u_{q_k}) < \delta. \tag{3.14}$$
From (SPbMS4), we have
$$\delta \leq d(u_{p_k}, u_{q_k}) \leq d(u_{p_k}, u_{p_{k-1}}) + \alpha d(u_{p_{k-1}}, u_{q_k}) - d(u_{p_{k-1}}, u_{p_{k-1}}). \tag{3.15}$$
As $k \to \infty$, applying upper and lower limit in inequation (3.15) and using equation (3.13) and inequation (3.14), we conclude
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_k}, u_{q_k}) \leq \lim_{k\to\infty} \sup d(u_{p_k}, u_{q_k}) \leq \alpha\delta. \tag{3.16}$$
Again from (SPbMS4), we have
$$\delta \leq d(u_{q_k}, u_{p_{k+1}}) \leq d(u_{q_k}, u_{q_{k+1}}) + \alpha d(u_{q_{k+1}}, u_{p_{k+1}}) - d(u_{q_{k+1}}, u_{q_{k+1}}). \tag{3.17}$$
As $k \to \infty$, applying upper and lower limit in inequation (3.17) and using equation (3.13) and inequation (3.14), we conclude
$$\delta \leq \lim_{k\to\infty} \inf d(u_{q_k}, u_{p_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{q_k}, u_{p_{k+1}}) \leq \alpha\delta. \tag{3.18}$$
By (SPbMS4), we have
$$\delta \leq d(u_{p_k}, u_{q_{k+1}}) \leq d(u_{p_k}, u_{p_{k-1}}) + \alpha d(u_{p_{k-1}}, u_{q_{k+1}}) - d(u_{p_{k-1}}, u_{p_{k-1}}). \tag{3.19}$$
As $k \to \infty$, applying upper and lower limit in inequation (3.19) and using equation (3.13) and inequation (3.14), we conclude
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_k}, u_{q_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{p_k}, u_{q_{k+1}}) \leq \alpha\delta. \tag{3.20}$$
By using (SPbMS4), we have
$$\delta \leq d(u_{p_{k+1}}, u_{q_{k+1}}) \leq d(u_{p_{k+1}}, u_{p_k}) + \alpha d(u_{p_k}, u_{q_{k+1}}) - d(u_{p_k}, u_{p_k}). \tag{3.21}$$
As $k \to \infty$, applying upper and lower limit in inequation (3.20), (3.21) and using equation (3.13) and inequation (3.14), we conclude
$$\delta \leq \lim_{k\to\infty} \inf d(u_{p_{k+1}}, u_{q_{k+1}}) \leq \lim_{k\to\infty} \sup d(u_{p_{k+1}}, u_{q_{k+1}}) \leq \alpha^2\delta. \tag{3.22}$$

Theorem 3.5. Let (E, d, α) be a complete SPbMS with parameter α and $S : E \to E$ be a continuous map. Suppose

1. there exists $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u, v \in E$ with $Su \neq Sv$,
$$F[\alpha^2 d(Su, Sv)] + \phi(d(u, v)) \leq F[M(u, v)], \tag{3.23}$$
where $M(u, v) = \max\{d(u, v), d(u, Su), d(v, Sv), d(v, Su)\}$,
2. for each sequence $\{a_n\} \in \mathbb{R}^+$, condition (ii) of Theorem (3.1) hold.

Then S has exactly one fixed point.

Proof. Define a sequence $\{u_n\}$ $\forall$ $n \in \mathbb{N}$, as follow, by using the point u_0 in E as an arbitrarily chosen point
$$Su_n = u_{n+1} = s^{n+1}u_0.$$
Assume that there is $p_0 \in \mathbb{N}$ such that $d(u_{p_0}, u_{p_0+1}) = 0$. Then by (SPbM2) $d(u_{p_0}, u_{p_0}) \leq d(u_{p_0}, u_{p_0+1})$ and $d(u_{p_0+1}, u_{p_0+1}) \leq d(u_{p_0}, u_{p_0+1})$. So, $d(u_{p_0}, u_{p_0}) = d(u_{p_0}, u_{p_0+1}) = d(u_{p_0+1}, u_{p_0+1})$. Thus, by (SPbM1) $u_{p_0} = u_{p_0+1}$, the proof is completed.
So, we assume that $d(u_n, u_{n+1}) > 0$ $\forall$ $n \in \mathbb{N}$.
From inequality (3.23), for all $n \in \mathbb{N}$, we get
$$F(d(Su_{n-1}, Su_n)) \leq F(\alpha^2 d(u_n, u_{n+1})) + \phi(d(u_{n-1}, u_n) \leq F(M(u_{n-1}, u_n), \tag{3.24}$$
where
$$M(u_{n-1}, u_n) = \max\{d(u_{n-1}, u_n), d(u_{n-1}, Su_{n-1}), d(u_n, Su_n), d(u_n, Su_{n-1})\}.$$
If $M(u_{n-1}, u_n) = d(u_n, u_{n+1})$, then
$$F(d(u_n, u_{n+1})) \leq F(\alpha^2 d(u_n, u_{n+1})) + \phi(d(u_{n-1}, u_n) \leq F(d(u_n, u_{n+1}),$$
that means
$$d(u_n, u_{n+1}) < d(u_{n-1}, u_n). \tag{3.25}$$
Here we get a contradiction, because $\phi(z) > 0$, $\forall$ $z > 0$. So, $M(u_{n-1}, u_n) = d(u_{n-1}, u_n)$. Thus from inequality (3.24),
$$F(d(u_n, u_{n+1})) \leq F(d(u_{n-1}, u_n)) - \phi(d(u_{n-1}, u_n)).$$
Repeating same process, we get
$$\begin{aligned} F(d(u_n, u_{n+1})) &\leq F(d(u_{n-1}, u_n)) - \phi(d(u_{n-1}, u_n)) \\ &\leq F(d(u_{n-2}, u_{n-1})) - \phi(d(u_{n-2}, u_{n-1})) - \phi(d(u_{n-1}, u_n)) \\ &\leq ... \leq F(d(u_0, u_1)) - \sum_{j=0}^{n} \phi(d(u_j, u_{j+1})). \end{aligned}$$
As $\liminf_{s\to t^+} \phi(s) > 0$, we have $\liminf_{n\to\infty} \phi(d(u_{n-1}, u_n)) > 0$. Now, according to definition of limit, there must be a number $N \in \mathbb{N}$ and $c_1 > 0$, such as for each $n > N$, $\phi(d(u_{n-1}, u_n)) > c_1$. So,
$$\begin{aligned} F(d(u_n, u_{n+1})) &\leq F(d(u_0, u_1)) - \sum_{j=0}^{N} \phi(d(u_j, u_{j+1})) - \sum_{j=N+1}^{n} \phi(d(u_j, u_{j+1})) \\ &\leq F(d(u_0, u_1)) - \sum_{j=N+1}^{n} c_1 \\ &= F(d(u_0, u_1)) - (n - N)c_1. \end{aligned}$$

As $n \to \infty$, we conclude

$$\lim_{n\to\infty} F(d(u_n, u_{n+1})) \leq \lim_{n\to\infty} [F(d(u_0, u_1)) - (n - N)].$$

This implies

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.26}$$

We now demonstrate that $\lim_{n\to\infty} d(u_n, u_{n+2}) = 0$. Suppose $d(u_n, u_p) > 0$ for each $n, p \in \mathbb{N}$. Otherwise, if we assume that for some $n = p + q$, where $q > 0$, $u_n = u_p$, then by using inequality (3.25), we conclude

$$d(u_p, u_{p+1}) = d(u_n, u_{n+1}) < d(u_{n-1}, u_n).$$

By taking same step again and again, we have

$$d(u_p, u_{p+1}) = d(u_n, u_{n+1}) < d(u_p, u_{p+1}).$$

Here we get a contradiction. So, $d(u_p, u_n) > 0$ for all $n, p \in \mathbb{N}$, where $n \neq p$.
Now, with the help of inequality (3.23) we conclude

$$\begin{aligned} F[d(u_n, u_{n+2})] &= F[d(Su_{n-1}, Su_{n+1})] \\ &\leq F[\alpha^2 d(Su_{n-1}, Su_{n+1})] \leq F[M(u_{n-1}, u_{n+1})] - \phi[d(u_{n-1}, u_{n+1})], \end{aligned}$$

where

$$M(u_{n-1}, u_{n+1}) = \max\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n), d(u_{n+1}, u_{n+2}), d(u_{n+1}, u_n)\}.$$

Therefore,

$$F[d(u_n, u_{n+2})] \leq F[\max\{d(u_{n-1}, u_{n+1}), d(u_{n-1}, u_n)\}] - \phi(d(u_{n-1}, u_{n+1})).$$

For our convenience, take $d_n^{'} = d(u_n, u_{n+2})$ and $d_n = d(u_n, u_{n+1})$. So,

$$Fd_n^{'} \leq F \max\{d_{n-1}^{'}, d_{n-1}\} - \phi(d_{n-1}^{'}). \tag{3.27}$$

Because of $(F1)$ condition of function F, we get

$$d_n^{'} \leq \max\{d_{n-1}^{'}, d_{n-1}\}.$$

Now, from inequation (3.25),

$$d_n \leq d_{n-1} \leq \max\{d_{n-1}^{'}, d_{n-1}\}.$$

That means for each $n \in \mathbb{N}$,

$$\max\{d_n^{'}, d_n\} \leq \max\{d_{n-1}^{'}, d_{n-1}\}.$$

Clearly, sequence $\max\{d_{n-1}^{'}, d_{n-1}\}$ is decreasing sequence of non negative real numbers. So, there must be a non negative real number β, so as

$$\lim_{n\to\infty} \max\{d_n^{'}, d_n\} = \beta.$$

Suppose $\beta > 0$, using equality (3.26), we conclude

$$\lim_{n\to\infty} \sup d_{n'} = \lim_{n\to\infty} \sup \max\{d_{n'}, d_n\} = \lim_{n\to\infty} \max\{d_{n'}, d_n\} = \beta.$$

By using continuity of F, and inequality (3.26), we obtain

$$\begin{aligned} F(\lim_{n\to\infty}) &\leq F(\lim_{n\to\infty} \sup \max\{d_{n-1}^{'}, d_{n-1}\}) - \lim_{n\to\infty} \sup \phi(d_{n-1}^{'}) \\ &< F(\lim_{n\to\infty} \sup \max\{d_{n-1}^{'}, d_{n-1}\}). \end{aligned}$$

Here we get a $F(\beta) < F(\beta)$,which is cntradiction. Hence

$$\lim_{n\to\infty} d(u_n, u_{n+2}) = 0. \tag{3.28}$$

After this, we have to prove that $\lim_{p,q\to\infty} d(u_p, u_q) = 0$ for each $p, q \in \mathbb{N}$. If it is not true then according to Lemma (3.1), there is a $\delta > 0$ and two sequences $\{p_k\}$ and $\{q_k\}$ such that

$$\lim_{k\to\infty} M(u_{p_k}, u_{q_k}) = \lim_{k\to\infty} \max\{d(u_{p_k}, u_{q_k}), d(u_{p_k}), u_{p_{k+1}}), d(u_{q_k}, u_{q_{k+1}}), d(u_{q_k}, u_{q_{k+1}})\} \leq \alpha\delta. \tag{3.29}$$

From inequation (3.23), we conclude

$$F[\alpha^2 d(u_{p_{k+1}}, u_{q_{k+1}})] \leq F[M(u_{p_k}, u_{q_k})] - \phi(d(u_{p_k}, u_{q_k})).$$

As $k \to \infty$, by using Lemma (3.1) and equality (3.29), we obtain

$$\begin{aligned} F\left[\frac{\delta}{\alpha}\alpha^2\right] &= F(\delta\alpha) \\ &\leq F[\alpha^2 \lim_{k\to\infty} \sup d(u_{p_{k+1}}, u_{q_{k+1}})] \\ &= \lim_{k\to\infty} \sup F[\alpha^2 d(u_{p_{k+1}}, u_{q_{k+1}})] \\ &\leq \lim_{k\to\infty} \sup F[M(u_{p_k}, u_{q_k})] - \lim_{k\to\infty} \sup \phi[d(u_{p_k}, u_{q_k})] \\ &= F[\lim_{k\to\infty} \sup M(u_{p_k}, u_{q_k})] - \lim_{k\to\infty} \sup \phi[d(u_{p_k}, u_{q_k})] \\ &\leq F[\lim_{k\to\infty} \sup M(u_{p_k}, u_{q_k})] - \lim_{k\to\infty} \inf \phi[d(u_{p_k}, u_{q_k})] \\ &< F[\lim_{k\to\infty} \sup M(u_{p_k}, u_{q_k})] \\ &< F(\alpha\delta). \end{aligned}$$

Because of $F1$ condition, we obtain $\alpha\delta < \alpha\delta$. This contradiction means $\lim_{p,q\to\infty} d(u_p, u_q) = 0$. Hence, $\{u_n\}$ is a Cauchy sequence. Because of completeness of (E, d), $\exists\ u^* \in E$, so that

$$\lim_{n\to\infty} d(u_n, u^*) = 0.$$

To demonstrate $Su^* = u^*$, we prove $d(Su^*, u^*) = 0$. Because if $d(Su^*, u^*) = 0$ then by (SPbM1) and (SPbM2) we can say $Su^* = u^*$. Let if possible $d(Su^*, u^*) > 0$. As $u_n \to u^*$ for $n \to \infty$, we have

$$\frac{1}{\alpha} d(u^*, Su^*) \leq \lim_{n\to\infty} \sup d(Su_n, Su^*) \leq \alpha d(u^*, Su^*). \tag{3.30}$$

From inequation (3.23) for each $n \in \mathbb{N}$, we obtain

$$F[\alpha^2 d(Su_n, Su^*)] \leq F[M(u_n, u^*)] - \phi(d(u_n, u^*))$$

Here

$$M(u_n, u^*) = \max\{d(u_n, u^*), d(u_n, Su_n), d(u^*, Su^*), d(u^*, Su_n)\}$$

and

$$\lim_{n\to\infty} \sup \max\{d(u_n, u^*), d(u_n, Su_n), d(u^*, Su^*), d(u^*, Su_n)\} = d(u^*, Su^*).$$

Taking $n \to \infty$, we get

$$\begin{aligned} F\left[\frac{1}{\alpha}\alpha^2 d(u^*, Su^*)\right] &= F(\alpha d(u^*, Su^*)) \\ &\leq F[\alpha^2 \lim_{k\to\infty} \sup d(Su_n, Su^*)] \\ &= \lim_{k\to\infty} \sup F[\alpha^2 d(Su_n, Su^*)] \\ &\leq \lim_{k\to\infty} \sup F[M(u_n, u^*)] - \lim_{k\to\infty} \phi[d(u_n, u^*)] \\ &= F[d(Su^*, u^*)] - \lim_{k\to\infty} \phi[d(u_n, u^*)] \\ &< F((Su_n, Su^*)). \end{aligned}$$

Now, because of $F1$ condition of function F, we have

$$\alpha d(u^*, Su^*) < d(u^*, Su^*),$$

which is a contradiction, because $\alpha \geq 1$. Now, it remains to prove that fixed point is unique. For this, we assume that there u^* and v^* are two different fixed point of S. Therefore, we obtain

$$d(u^*, v^*) = d(Su^*, Sv^*) > 0.$$

By inequation (3.23),

$$F[d(u^*,v^*)] = F[d(Su^*,Sv^*)] \leq F[\alpha^2 d(Su^*,Sv^*)] \leq F[M(u^*,v^*)] - \phi(d(u^*,v^*)).$$

Here,

$$M(u^*,v^*) = \max\{d(u^*,v^*), d(u^*,Su^*), (v^*,Sv^*), d(Su^*,v^*)\} = d(u^*,v^*).$$

Thus, we get

$$F[d(u^*,v^*)] \leq F[d(u^*,v^*)] - \phi(d(u^*,v^*)) < F[d(u^*,v^*)]$$

That means $d(u^*,v^*) < d(u^*,v^*)$, which is not true. So, $u^* = v^*$. This completes the proof.

Corollary 3.6. "Let (E,d,α) be a complete SPbMS with parameter α and $S : E \to E$ be a continuous map. Suppose

1. there exists $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u,v \in E$ with $Su \neq Sv$,
$$F[\alpha^2 d(Su,Sv)] + \phi(d(u,v)) \leq F\left[\frac{d(u,Su)+d(v,Sv)}{2}\right],$$
2. for each sequence $\{a_n\} \in \mathbb{R}^+$, condtion (ii) of Theorem (3.1) hold.

Then S has exactly one fixed point."

Proof. It is easy to demonstrate, because,

$$\begin{aligned} F[\alpha^2 d(Su,Sv)] + \phi(d(u,v)) &\leq F\left[\frac{d(u,Su)+d(v,Sv)}{2}\right] \\ &\leq F[\max\{d(u,Su), d(v,Sv)\}] \\ &\leq F[\max\{d(u,v), d(u,Su), d(v,Sv), d(Su,v)\}]. \end{aligned}$$

Corollary 3.7. "Let (E,d,α) be a complete SPbMS with parameter α and $S : E \to E$ be a continuous map. Suppose

1. there exists $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for any $u,v \in E$ with $Su \neq Sv$,
$$F[\alpha^2 d(Su,Sv)] + \phi(d(u,v)) \leq F\left[\frac{d(u,v)+d(u,Su)+d(v,Sv)}{3}\right],$$
2. for each sequence $\{a_n\} \in \mathbb{R}^+$, condtion (ii) of Theorem (3.1) hold.

Then S has exactly one fixed point."

Proof. It is easy to demonstrate, because,

$$\begin{aligned} F[\alpha^2 d(Su,Sv)] + \phi(d(u,v)) &\leq F\left[\frac{d(u,v)+d(u,Su)+d(v,Sv)}{3}\right] \\ &\leq F[\max\{d(u,v), d(u,Su), d(v,Sv), d(Su,v)\}]. \end{aligned}$$

4 Conclusion

Here, we provided some fixed point findings for nonlinear F-type contractions in Strong Partial b-Metric Spaces (SPbMS). We also included some examples that demonstrates the applicability of our findings.

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