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A Study on The Gamma Graph of Cycle C_{3K+1}

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Abstract

For a graph $G = (V,E)$, a set S $\subseteq V$ is a dominating set if every vertex in $V-S$ is adjacent to at least one vertex in S.The domination number $\gamma(G)$ of G equals the minimum cardinality of a dominating set in G. A dominating set is aγ -set in G if $|S = \gamma(G)|$. A gamma graph $\gamma.G$ of a garph G is a graph with S as a vertex set, if S_1 S_2 are adjacent iff there exist two vertices u and v of G suchthat . In this paper we initiate the study of gamma graph of cycle C_{3k+1}

Key words: Dominating Set, Gamma Set, Gamma Graph

AMS classification: 05C38, 05C69

1 Introduction

We consider only finite simple graphs $G = (V,E)$. We use standard notations of graph theory, as in Balakrishnan and Ranganathan [2]. For an introduction to the theory of domination in graphs we refer to Haynes et al. [8] A set S $\subseteq v$of vertices in a graph $G = (V,E)$ is calleda dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. The domination number $\gamma(G)$ of G equals the minimum cardinality of a dominating set S in G and the set S is known as $\gamma - set$.

The concept of the gamma graph is introduced by Sridharan and Subramanian[9]. The γ graph of G, denoted by γG and defined as the graph with vertex set S where S is the collection of all $\gamma-$sets in a graph G and any two vertices s_1 and s_2 are adjacent if $|S_1 \cap S_2| = \gamma(G) - 1$. In 2011 G. H. Fricke et al. [7] inadvertently defined gamma graphs as $G(\gamma) = (V(\gamma), E(\gamma))$ of G to be the graph whose vertices $v(\gamma)$ corresponds 1-to-1 with the γ- sets of G, and two γ- sets, say S_1 and S_2 are adjacent in $E(\gamma)$ if there exists a vertex $V \in S_1$ and a vertex $W \in S_2$ such that v is adjacent to W and $S_1 = S_2 - \{w\} \cup \{v\}$ or equivalently, $S_2 = S_2 - \{v\} \cup \{w\}$. Note that both the definitions of gamma graphs are different from each other.Throughout the paper, we use the definition of gamma graphs given by Subramanian and Sridharan.

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Many researchers have studied on gamma graphs. Gamma graphs of some special classes of trees are studied by Bien [3]. Modified γ graph- $G(\gamma_m)$ of some grid graphs is studied by Anusuya and Kala [1] while Sridharan et al. [10] discussed induced subgraph of gamma graphs and they used the definition provided in [9]. Also the gamma graphs of trees are studied by Finbow and Bommel [6]. Connelly et al. [5] have investigated when the gamma graph is disconnected. Bommel [4] is given the note on bipartite graph that is not the gamma graph of a bipartite graph in the context of definition provided in [7]. Some gamma graphs of cycles were determined by G. H. Fricke et al. [7] and that is, for $k \geq 2, C_{3k}(\gamma) \cong \overline{K}_3$and $C_{3k+2}(\gamma) \cong C_{3k+2}$.As mentioned in [3] every graph G (γ) is a spanning subgraph of $\gamma.G$. These existing results in the literature inspire us to characterize the order of $\gamma.C_{3k+1}$.

2. Construction of Gamma Graph of Cycle C_{3k+1}

We discuss the induction method of obtaining gamma graphs of C_{3k+1}.

Method of obtaining gamma sets for C_{3k+1}

Let C_4 be the cycle with vertices $v_1, v_2 v_3$ andv_4 since $\gamma(C_4) = \lceil \frac{4}{3} \rceil$,The $\gamma-$sets of C_4 will be of cardinality 2. The gamma sets of C_4 can be $S_1 = v_1, v_2, S_2 = v_2, v_3, S_3 = v_3, v_4,\ S_4 = v_1, v_4, S_5 = v_2, v_4, S_6 = v_1, v_3$.These $\gamma-$sets will be the vertices of $\gamma.C_4$ by defitions of gamma graphs. Hence $|V(\gamma.C_4)| = 6$

Since $\gamma(C_7) = \lceil \frac{7}{3} \rceil = 3$,We need $\gamma-$sets with cardinality 3.We obtain $\gamma-$sets of C_7 from $\gamma.C_4$ by the following procedure:

1. Include v_5,v_6 andv_7 in S_1, S_2 andS_3 respectively,We got distinct gamma sets of order 3 say,$P_1 = v_1, v_2, v_5$,$P_2 = v_2, v_3, v_6$,$P_3 = v_3, v_4, v_7$.
2. By keeping the vertex v_5 and one of its adjacent vertex fix in C_7 we get $P_4 = \{v_1, v_4, v_5\}$ and $P_5 = \{v_2, v_5, v_6\}$ respectively. Similarly, by keeping the vertex v_6 and its adjacent vertex fix in C_7 we get$P_7 = \{v_1, v_4, v_7\}$.
3. Fix non adjacent vertices of above γ-sets P_i;$(i = 1, 2, ..., 7)$ to obtain other $\gamma-sets$. Thus the $\gamma-$setsare $P_8 = v_1, v_3, v_5$,$P_9 = v_1, v_3, v_6$,$P_{10} = v_2, v_4, v_6$,$P_{10} = v_2, v_4, v_6$,$P_{11} = v_2, v_4, v_7$,$P_{12} = v_1, v_4, v_6$,$P_{13} = v_2, v_5, v_7$ and $P_{14} = v_3, v_5, v_7$ respectively.

Thus ,we get 14γ-sets with cardinally 3 and hence, $|V(\gamma.C_7)| = 14$.

since $\gamma(C_{10}) = 4$ we obtain γ-sets of C_{10} with cardinality 4 by the following ways:

1. Include v_8,v_9 and v_{10}in P_1,P_2 and P_3 as well as in P_4,P_5 and P_6respectively,We

get gamma sets of order 4 say, $T_1 = v_1, v_2, v_5, v_8$,$T_2 = v_2, v_3, v_6, v_9$,$T_3 = v_3, v_4, v_7, v_{10}$,$T_4 = v_1, v_4, v_5, v_8$,$T_5 = v_2, v_5, v_6, v_9$,$T_6 = v_3, v_6, v_7, v_{10}$.

2. In C_{10} keep the vertex v_8 and one of its adjacent vertices fix we get $T_7 = \{v_1, v_4, v_7, v_8\}$ and $T_8 = \{v_2, v_5, v_8, v_9\}$ respectively. By keeping the vertex v_9 and v_{10} fix we get $T_9 = \{v_3, v_6, v_9, v_{10}\}$, Similarly, keep the vertex v_{10} and v_1 fix we get$T_{10} = \{v_1, v_4, v_7, v_{10}\}$,
3. Fix any three non-adjacent vertices of above γ-sets (T_i) to obtain other γ-sets. Thus the γ-sets are $T_{11} = v_1, v_3, v_5, v_8$,$T_{12} = v_3, v_5, v_8, v_{10}$,$T_{13} = v_2, v_4, v_6, v_9$,$T_{14} = v_1, v_3, v_6, v_9$,$T_{15} = v_2, v_4, v_7, v_{10}$,$T_{16} = v_3, v_5, v_7, v_{10}$,$T_{17} = v_1, v_4, v_6, v_8$,$T_{18} = v_2, v_5, v_8, v_{10}$,$T_{19} = v_3, v_6, v_8, v_{10}$,$T_{20} = v_1, v_4, v_7, v_9$ and $T_{21} = v_2, v_5, v_8, v_{10}$.
4. Fix any two non-adjacent vertices of above γ-sets (T_i) to obtain other γ-set. Thus the $\gamma - sets$ are $T_{22} = v_2, v_4, v_7, v_9$,$T_{23} = v_1, v_4, v_6, v_9$,$T_{24} = v_1, v_3, v_6, v_8$ and $T_{25} = v_2, v_5, v_7, v_{10}$.

   Thus , we get 25γ-sets of C_{10} with cardinality 4. so,$|V(\gamma.C_{10})| = 25$.

   Proceeding in this way we can obtain γ-sets for C_{3k+1}.

Theorem 2.1 $|V(\gamma.C_{3k+1})| = \frac{(3k+1)(k+2)}{2}; k \geq 1$.

proof:Let $v_1, v_2, ..., v_{3k+1}$ be the vertex set of C_{3k+1}. Since $\gamma(C_{3k+1}) = \lceil \frac{3k+1}{3} \rceil = k+1$ we obtain γ-sets of C_{3k+1} with cardinality $k+1$,

As per the procedure mentioned earlier, we can obtain γ-sets for C_{3k+1} in the following ways:

1. With one adjacent pair of vertices; As the cycle is of length $3k+1$.
2. with alternate pair of vertices; As there are $3k+1$ choice for the first pair of alternate vertices and each γ-sets have k choices for other pair where some γ-sets occurs twice.So, the total number of γ-sets with alternate vertices is $\frac{(3k+1)k}{2}$.

Hence, the no of γ-sets for

$$\begin{aligned} C_{3k+1} &= (3k+1) + \frac{(3k+1)k}{2} \\ &= \frac{(6k+2)+(3k^2+k)}{2} \\ &= \frac{3k^2+7k+2}{2} \\ &= \frac{(k+2)(3k+1)}{2}. \end{aligned}$$

Thus,$|V(\gamma.C_{3k+1})| = \frac{(k+2)(3k+1)}{2}; k \geq 1$.

Theorem 2.2 The gamma graph of C_{3k+1} is 4-regular.
proof.Let $v_1, v_2, ..., v_{3k+1}$ be the vertex set of cycle C_{3k+1}.By the method of obtaing γ-sets we observe:

1. The set $S_1 = v_1, v_4, v_7, ..., v_{3k-2}, v_{3k+1}$ is the γ-set with one pair of adjacent vertices.
2. The set $S_1 = v_1, v_2, v_6, ..., v_{3k-3}, v_{3k+1}$ is the γ-set with alternate pair of vertices of 4.
3. The γ-set S_1 is adjacent to

$$s_3 = \{v_1, v_4, v_7, ..., v_{3k-2}, v_{3k-1}\},$$
$$s_4 = \{v_3, v_4, v_7, ..., v_{3k-2}, v_{3k+1}\},$$
$$s_5 = \{v_2, v_4, v_7, ..., v_{3k-2}, v_{3k+1}\},$$
$$s_6 = \{v_1, v_4, v_7, ..., v_{3k-2}, v_{3k}\}$$

4. The γ-set S_2 is adjacent to

$$s_7 = \{v_1, v_3, v_6, ..., v_{3k-3}, v_{3k}\},$$
$$s_8 = \{v_1, v_3, v_5, ..., v_{3k-3}, v_{3k-1}\},$$
$$s_9 = \{v_3, v_6, v_7, ..., v_{3k-1}, v_{3k+1}\},$$
$$s_{10} = \{v_1, v_4, v_6, ..., v_{3k-3}, v_{3k-1}\}.$$

In this way each γ-set is adjacent with4other γ-sets. Hence $\gamma.C_{3k+1}$ is $4-$regular.

3 Conclusion

Gamma graph of C_{3k+2} is isomorphic to itself and that of C_{3k} is isomorphic to complement of K_3. We have taken the initiative to study on the nature of gamma graph of cycle C_{3k+1} and observed that it is a regular graph.

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