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**Research Article**

# Sub-Restrained Perfect Domination In Graphs

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**ARTICLE INFO** **ABSTRACT**

Let $G = (V(G), E(G))$ be a connected graph. Let $M \subseteq V(G)$ be a minimum perfect dominating set and $T \subseteq V(G)\backslash M$ is said to be sub-restrained perfect dominating set of $G$ if every $v \in V(G) \setminus T$ such that $|N(v) \cap T| = 1$. The sub-restrained perfect dominating number of $G$ is the minimum cardinality of the sub-restrained perfect dominating set of $G$ which is denoted by $\gamma_{srp}(\mathrm{G})$. As a novel approach to the study of domination theory, we can attempt to define sub-restrained perfect dominating set in this paper. We also identify certain novel findings, fundamental characteristics, and so forth.

**Keywords:** Dominating set, perfect dominating set, sub-restrained perfect dominating set, restrained dominating set, corona product of two graphs.

## 1 Introduction:

By a graph $G$, We means a finite connected and undirected graph without loops and parallel edges with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, the open neighbourhood of $v$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the close neighbourhood of $v$ is $N[v] = N(v) \cup v$. A set $M \subseteq V(G)$ is dominating set of G if every $v \in V(G)\backslash M$ such that $m \in M, vm \in E(G)$. The minimum cardinality of dominating set $M$ is called domination number denoted by $\gamma(G)$[10]. A dominating set $M$ is of the smallest possible size in graph $G$ is called minimum dominating set[12]. A subset $M \subseteq V(G)$ is perfect dominating set if each vertex $v \in V(G) \setminus M$ is adjacent to exactely one vertex in $M$. The minimum cardinality of perfect dominating set $M$ is called perfect domination number denoted by $\gamma_p(G)$[1]. A dominating set $M$ is a restrained dominating set of $G$ if for every $v \in V(G) \setminus M$, $\exists\, z \in V(G) \setminus M$ such that $vz \in E(G)$. A restrained perfect dominating set of $G$ is a subset $M \subseteq V(G)$ such that $M$ is a dominating set of $G, M = V(G)$ or $< V(G) \setminus M >$ has no isolated vertices and for every $v \in V(G) \setminus M, |N(v) \cap M| = 1$. The minimum cardinality of restrained perfect dominating set $M$ is called restrained perfect domination number denoted by $\gamma_{rp}(G)$[1].
If $M \subseteq V(G)$ be a minimum perfect dominating set and $T \subseteq V(G)\backslash M$ is said to be sub-restrained perfect dominating set of $G$ if every $v \in V(G) \setminus T$ such that $|N(v) \cap T| = 1$. The sub-restrained perfect dominating number of $G$ is the minimum cardinality of the sub-restrained perfect dominating set of $G$ which is denoted by $\gamma_{srp}(\mathrm{G})$. Keep in mind that the sub-restrained perfect domination number, which is the lowest cardinality of the sub-restrained perfect dominating set, is represented by $\gamma_{srp}(G)$.

## 2 Main Results:

**Theorem 1. Let $G = P_n$ then**

$$\gamma_{srp}(P_n) = \begin{cases} \frac{n+2}{3} & ; \text{if } n = 3k+1; k \in \mathbb{N} \\ \frac{n+1}{3} & ; \text{if } n = 3k+2; k \in \mathbb{N} \cup \{0\} \end{cases}$$

**Proof:** Let $G = P_n$ be a graph with $n$-vertices namely $x_1, x_2, \dots, x_n$ and $V(G)$ is a vertex set of graph $G$.
**Case: 1 If $n = 3k+1, k \in \mathbb{N}$**

Let $M_1$ be the minimal perfect dominating set of graph $G$ and $M_1 = \{x_1, x_4, x_7 \dots, x_{3k+1}\}$.
So, $M_1$ contains $k$+1 number of vertices.

Now, $V(G) \setminus M_1 = \{x_2, x_3, x_5, x_6, \ldots, x_{3k-1}, x_{3k}\}$ and the cardinality of $V(G) \setminus M_1$ is given by
$|V(G) \setminus M_1| = n - (k+1) = 3k + 1 - k - 1 = 2k$.
Let $T_1$ be the sub-restrained perfect dominating set.
Consequently, by definition, we may state that $T_1 \subseteq V(G) \setminus M_1$
Thus, $T_1 = \{x_2, x_5, x_8, \ldots, x_{3k-1}, x_{3k}\}$

$$|T_1| = k + 1$$
$$= \frac{n-1}{3} + 1$$
$$= \frac{n-1+3}{3}$$
$$|T_1| = \frac{n+2}{3}$$
$$\therefore \gamma_{srp}(G) = \frac{n+2}{3}$$

$\therefore G = P_n$ with $n = 3k + 1$ vertices has $\frac{n+2}{3}$ vertices in its sub-restrained perfect dominating set.

**Case : 2 If $n = 3k + 2, k \in \mathbb{N} \cup \{0\}$**
Let $M_2$ be the minimal perfect dominating set of graph $G$ and $M_2 = \{x_1, x_4, x_7, \ldots, x_{3k+1}\}$ So, $M_2$ contains $k + 1$ vertices.
Now, $V(G) \setminus M_2 = \{x_2, x_3, x_5, x_6, \ldots, x_{3q}, x_{3q+2}\}$ and the cardinality of $V(G) \setminus M_2$ is given by

$$|V(G) \setminus M_2| = n - (k+1)$$
$$= 3k + 2 - k - 1$$
$$= 2k + 1$$

Let us consider $T_2$ be the sub-restrained perfect dominating set. Consequently, by definition, we may state that $T_2 \subseteq V(G) \setminus M_2$.
Thus, $T_2 = \{x_2, x_5, x_8, \ldots, x_{3q+2}\}$

$$|T_2| = k + 1$$
$$= \frac{n-2}{3} + 1$$
$$= \frac{n-2+3}{3}$$
$$|T_2| = \frac{n+1}{3}$$
$$\therefore \gamma_{srp}(G) = \frac{n+1}{3}$$

$\therefore G = P_n$ with $n = 3q + 2$ vertices has $\frac{n+1}{3}$ vertices in its sub-restrained perfect dominating set.
Thus, demonstrate the outcome.

**Corollary 1. Let $G = P_n$ where $n = 3k, (k \in \mathbb{N})$ then $\gamma_{srp}(G)$ does not exist.**

**Proof:** Let $G = P_n$ be a graph with $n = 3k, (k \in \mathbb{N})$ vertices namely $x_1, x_2, \ldots, x_n$ and $V(G)$ is a set of vertices of graph $G$.
Now, let $M$ be a minimal perfect dominating set.
$\therefore M = \{x_1, x_2, \ldots, x_{3k-2}, x_{3k-1}\}$ and $V(G) \setminus M = \{x_2, x_3, x_5, x_6, \ldots, x_{3k}\}$.
We now need to identify the dominating $T$ such that $T \subseteq V(G) \setminus M$ and $T$ is a sub-restrained perfect dominating set.
Therefore $T = \{x_2, x_5, \ldots, x_{3k}\}$.
In this instance, it is evident that $x_{3k-2}$ is not dominating by any vertices in $V(G) \setminus M$.
$\therefore$ No sub-restrained perfect dominating set is known to exist for $P_n$ where $n = 3k$.
So present the result.

**Theorem 2. Let $G = C_n$ then**

$$\gamma_{srp}(C_n) = \begin{cases} \frac{n}{3} & ; \text{if } n = 3k; k \in \mathbb{N} \\ \frac{n+2}{3} & ; \text{if } n = 3k+1; k \in \mathbb{N} \end{cases}$$

**Proof:** Let $G = C_n$ be a graph with $n$-vertices namely $x_1, x_2, \ldots, x_n$ and $V(G)$ is a vertex set of graph $G$.
**Case: 1 If $n = 3k, k \in \mathbb{N}$**
Let $M_1$ represent minimal perfect dominating set of graph $G$.
Consider $M_1 = \{x_1, x_4, x_7, \ldots, x_{3k-2}\}$, it contains $k$ number of vertices.

Now $V(G) \setminus M_1 = \{x_2, x_3, x_5, x_6, \ldots, x_{3k-3}, x_{3k-1}, x_{3k}\}$ and the cardinality of $V(G) \setminus M_1$ is given by

$$|V(G) \setminus M_1| = n - k = 3k - k = 2k$$

Let $T_1$ be the sub-restrained perfect dominating set of graph $G$.
Thus, by definition, we may say that $T_1 \subseteq V(G) \setminus M_1$.
As a result, set $T_1's$ cardinality is provided by

$$|T_1| = k = \frac{n}{3} = \gamma_{srp}(G)$$

$\therefore G = C_n$ with $n = 3k + 1$ vertices has $\frac{n}{3}$ vertices in its sub-restrained perfect dominating set.

**Case : 2 If $n = 3k + 1, k \in \mathbb{N}$**

Let $M_2$ be the minimal perfect dominating set of graph $G$.
Now we assert that $M_2 = \{x_1, x_4, x_7, \ldots, x_{3k-2}, x_{3k-1}\}$.
$V(G) \setminus M_2 = \{x_2, x_3, x_5, x_6, \ldots x_{3k-3}, x_{3k}, x_{3k+1}\}$ and the cardinality of $V(G) \setminus M_1$ is given by

$$|V(G) \setminus M_2| = n - (k + 1) = 3k + 1 - (k + 1) = 3k + 1 - k - 1 = 2k$$

By definition, if $T_2$ is the sub-restrained perfect dominating set, we can state that $T_2 \subseteq V(G) \setminus M_2$.

Thus, $T_2 = \{x_2, x_5, x_8, \ldots, x_{3k-3}, x_{3k}\}$

$$|T_2| = k + 1 = \frac{n-1}{3} + 1 = \frac{n - 2 + 3}{3}$$

$$\therefore \gamma_{srp}(G) = \frac{n+2}{3}$$

$\therefore G = P_n$ with $n = 3k + 1$ vertices has $\frac{n+2}{3}$ vertices in its sub-restrained perfect dominating set.

**Corollary 2. Let $G = C_n$ where $n = 6k - 1$ and $n = 6k + 2, (q \in \mathbb{N})$ then $\gamma_{srp}(G)$ does not exist.**

**Proof: Case: 1 If $n = 6k - 1, k \in \mathbb{N}$**

Let $G = P_n$ be a graph with $n = 6k - 1, (k \in \mathbb{N})$ vertices namely $x_1, x_2, \ldots, x_n$. Moreover $V(G)$ is a set of vertices of graph $G$.
Let $M_1$ be a minimal perfect dominating set.
$\therefore M_1 = \{x_1, x_4, \ldots, x_{6k-2}, x_{6k-1}\}$ and $V(G) \setminus M_1 = \{x_2, x_3, x_5, x_6, \ldots, x_{6k-4}, x_{6k-3}\}$.
We now need to identify the dominating set $T_1$ such that $T_1 \subseteq V(G) \setminus M_1$ and $T_1$ is a sub-restrained perfect dominating set.
Now, we assert that $T_1 = \{x_2, x_5, \ldots, x_{6k-4}, x_{6k-3}\}$.
In this instance, it is evident that vertex $x_{6k-1}$ is not dominated by any vertex form the set of vertices $V(G) \setminus M_1$.
$\therefore$ No sub-restrained perfect dominating set is known to exist for $C_n$ where $n = 6k - 1$.

**Case: 2 If $n = 6k + 2, q \in \mathbb{N}$**

Let $G = P_n$ be a graph with $n = 6k + 2, (q \in \mathbb{N})$ vertices namely $x_1, x_2, \ldots, x_n$. And $V(G)$ is a set of vertices of graph $G$.
Let $M_2$ now represent a minimal perfect dominating set.
$\therefore M_2 = \{x_1, x_4, x_7, \ldots, x_{6k+1}, x_{6k+2}\}$ and $V(G) \setminus M_2 = \{x_2, x_3, x_5, x_6, \ldots, x_{6k-1}, x_{6k}\}$.
Finding a dominating set $T_2$, that is both a sub-restrained perfect dominating set and such that $T_2 \subseteq V(G) \setminus M_2$ is our current goal.
Now we assert that $T_2 = \{x_2, x_5, x_8, \ldots, x_{6k-1}, x_{6k}\}$.
Here, we can clearly observe that vertex $x_{6k+2}$ is not dominating by any vertices in $V(G) \setminus M_2$. As, $x_{6k}$ can dominate two vertices $x_{6k+1}, x_{6k-1}$ and $x_2$ can dominate to vertices which are $x_1, x_3$.
$\therefore$ There does not exist any sub-restrained perfect dominating set for $C_n$ where $n = 6k + 2$.

**Corollary 3.** If $n \geq 2$ be a positive integer, $\gamma_{srp}(K_n) = 1$.

**Proof:** Let $G$ be a complete graph with $n$ vertices namely $x_1, x_2, \dots, x_n$.
As, $G$ is complete graph each vertices of $G$ is adjacent to every other vertex of graph $G$.
i.e. $d(v_i) = n - 1\,; i = 1,2, \dots n$
So, the minimal dominating set of $G$ is $M = \{x_k\}\,; 1 \le k \le n$
And $V(G)\setminus M = \{x_1, x_2, \dots, x_n\}$
Let $T$ be the sub-restrained perfect dominating set of $G$ and $T \subseteq V(G)\setminus M$.
$T = \{x_l\}\,; 1 \le l \le n$ and $l \ne k$.
$\therefore \gamma_{srp}(K_n) = 1$.

**Corollary 4.** Let $G$ be any graph which is connected and complete then $\gamma_{srp}(GoH) = |V(G)|$. (Where $GoH$ denotes the corona product of two graph G and H.)

**Theorem 3. Let $G$ be a connected graph of order $n \ge 4$ then $1 \le \gamma_{srp}(G) < n$.**

**Proof:** Let $G$ be connected graph with $n$ vertices namely $\{x_1, x_2, \dots, x_n\}$.

**Case-1: Conversely assert that $\gamma_{srp}(G) \ge n$.**
Assume that $T_1$ is a sub-restrained perfect dominating set of $G$ then it is clear that $|T_1| \le n$ and $|T_1| \subseteq V(G) \setminus M_1$ (by definition), where $M_1$ is a minimal dominating set of graph $G$.
i.e. $V(G) \setminus M_1 \le n$.
But $G$ is a graph with $n$ vertices, i.e. $|V(G)| = n$. And hence the cardinality of set $|V(G) \setminus M|$ is must be more than $n$. It is contradict with our assertion. That's why $|V(G) \setminus M| \le n$ is not possible.
$\therefore \gamma_{srp}(G) < n$

**Case-2: Conversely assert that $\gamma_{srp}(G) < 1$**
Assume that $T_2$ is a sub-restrained perfect dominating set of $G$.
Then $|T_2| < 1$
so, the only possibility for $T_2$ is zero.
i.e. $|T_2| = 0$
which is not possible as $G$ is connected graph.
$\therefore \gamma_{srp}(G) \ge 1$

**Theorem 4. Let $G$ be a connected graph of order $n \ge 4$ then $M \cap T = \phi$ where $M$ is $\gamma_p$ - set and $T$ is a $\gamma_{srp}$ - set of $G$.**
**Proof:** Let $G$ be a connected graph of order $n$ say $\{x_1, x_2, \dots x_n\}$ and given $M$ is a minimal perfect dominating set of $G$ and $T$ is a sub-restrained perfect dominating set of $G$.
i.e. by definition $T \subseteq V(G) \setminus M$ where $V(G)$ is a vertex of $G$.
$\therefore M \cap T \subseteq M \cap V(G) \setminus M$
and
$M \cap V(G) \setminus M = \phi$
$\therefore M \cap T = \phi$

**Theorem 5. Let $G$ and $H$ be connected graph and $\gamma(G) = 1 = \gamma(H)$ then $\gamma_{srp}(G + H) = 1$.**
**Proof:** Let $T_1$ and $T_2$ be a the dominating set of graph $G$ and $H$ respectively.
i.e., $|T_1| = 1$ and $|T_2| = 2$
suppose, $T_1 = \{a\}$ and $T_2 = \{b\}$
Now, $G = T_1 + P_1\,;$ where $V(P_1) = V(G) \setminus V(T_1)$
and $H = T_2 + P_2\,;$ where $V(P_2) = V(H) \setminus V(T_2)$
Thus, $G + H = (T_1 + P_1) + (T_2 + P_2) = T_1 + (T_2 + P_1 + P_2)$
$\gamma_{srp}(G + H) = |T_1| = 1$.

**Corollary 5. Let $G$ be a connected graph of order $n \ge 2$ then $\gamma_{srp}(G) = 1$ if and only if $G = P + H$ where $P, H \subseteq G$.**
(Proof mentioned as above.)

**Corollary 6. Let $G$ be a connected graph of order $n \ge 3$ then $\gamma_{srp}(G) = 1$ if and only if $G = Q + H$ where $Q, H \subseteq G$.**
(Proof mentioned as above.)

## 3 Conclusion:

We introduced a new type of domination called sub-restrained domination and successfully calculated its number for various graphs and obtained specific properties. This approach enriches our knowledge and offers exciting prospects for comparable results for different domination models and graph operations, advancing our understanding of graph theory.

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