On Kasaj-b-open Sets and Kasaj-regular-open Sets in Kasaj Topological Spaces

[1]Kashyap G. Rachchh, [2]Sajeed I. Ghanchi, [3]Asfak A. Soneji

[1]Research Scholar, Department of Mathematics, Institute of Infrastructure, Technology, Research and Management (IITRAM), Ahmedabad, India

[2]Assi. Professor, Department of Mathematics, Atmiya University, Rajkot, India

[3]PG student, Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, India.

Abstract : Recently, We defined Kasaj-topological space and weak open sets namely Kasaj-pre-open sets, Kasaj-semi-open sets, Kasaj-alpha-open sets, Kasaj-beta-open sets in Kasaj topological spaces and analyzed their basic properties. Also we defined new types of continuous functions namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function, Kasaj-alpha-continuous function and Kasaj-beta-continuous function in Kasaj topological spaces. The purpose of this paper is to define new classes of sets called Kasaj-b-open(KS-b-open) sets, Kasaj-regular-open(KS-regular-open) sets and a new types of continuous functions namely Kasaj-b-continuous function in Kasaj topological spaces. We shall investigate their basic properties and also analyze relation among Kasaj-open sets, Kasaj-pre-open sets, Kasaj-semi-open sets, Kasaj-alpha-open sets, Kasaj-beta-open sets, Kasaj-b-open sets and Kasaj-regular-open sets.

Index Terms **- Kasaj topological space, Kasaj-b-open set, Kasaj-regular-open set.**

I. Introduction and Preliminary

In recent times many people have introduced new topological space and it is studied very well. For example, nano topological space was introduced by L. Thivagar et al. **[3]**. S. Chandrasekar **[5]** introduced Micro topological spaces which are extension of nano topological spaces. He has used Levine's simple extension concepts in nano topological spaces. The notations of Semi-open sets, Pre-open sets, α-open sets, β-open sets, **b**-open sets and regular-open sets were introduced by Levine **[4]** , Mashhour et al. **[1]**, M. Caldas **[6]**, M. E. Abd El-Monsef et al. **[7]**, D. Andrijevic **[9]** and K.F. porter **[10]** respectively. We defined and studied **[8]** Kasaj-topological space and weak open sets namely Kasaj-pre-open sets and Kasaj-semi-open sets in Kasaj topological spaces, also we defined **[11]** Kasaj-α-open sets and Kasaj-β-open sets in Kasaj topological spaces and analyze their basic properties. Also we defined new types of continuous functions namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function, Kasaj-α-continuous function and Kasaj-β-continuous function in Kasaj topological space. In this paper, we shall define new classes of sets on Kasaj topological spaces namely Kasaj-b-open (KS-b-open) sets and Kasaj-regular-open (KS-regular-open) sets and find the relation of these new classes with existing classes.

Definition : 1 A subset P of a topological space (X, τ) is called

- a semi-open set **[4]** if $P \subset cl(int(P))$.
- a pre-open set **[1]** if $P \subset int(cl(P))$.
- a α-open set **[6]** if $P \subset int(cl(int(P)))$.
- a β-open set **[7]** if $P \subset cl(int(cl(P)))$.
- a b-open set **[9]** if $P \subset int(cl(P)) \cup cl(int(P))$.
- A regular-open set **[10]** if $A = int(cl(P))$.

The complement of a semi-open set (pre-open set, α-open set, β-open set, b-open set, regular-open set) in a space X is called semi-closed set (pre-closed set, α-closed set, β-closed set, b-closed set, regular-closed set) in X.

II. Nano Topological Spaces

Definition: 2 Let Ω be a non-empty Universal set and $\mathfrak{R}$ be an equivalence relation on Ω and it is named as *the indiscernibility relation*. As $\mathfrak{R}$ is an equivalence relation, we get disjoint partition and each equivalence class is indiscernible with one another. The pair $(\Omega, \mathfrak{R})$ is called as *approximation space*. Let $X \subset \Omega$.

- *The lower approximation* of X with respect to $\mathfrak{R}$ is denoted by $\mathscr{L}_{\mathfrak{R}}(X)$ and is defined by $\mathscr{L}_{\mathfrak{R}}(X) = \cup_{x\in\Omega}\{P(x) : P(x) \subset X\}$ where P(x) denotes the equivalence relation which contains $x\in\Omega$.
- The upper approximation of X with respect to $\mathfrak{R}$ is denoted by $\mathbf{U}_{\mathfrak{R}}(X)$ and is defined by $\mathbf{U}_{\mathfrak{R}}(X) = \cup_{x\in\Omega}\{P(x) : P(x) \cap X \neq \emptyset\}$ where P(x) denotes the equivalence relation which contains $x\in\Omega$.
- The boundary region of X with respect to $\mathfrak{R}$ is denoted by $\mathscr{B}_{\mathfrak{R}}(X)$ and is defined by $\mathscr{B}_{\mathfrak{R}}(X) = \mathbf{U}_{\mathfrak{R}}(X) - \mathscr{L}_{\mathfrak{R}}(X)$.

Definition: 3 Let Ω be an universal set. $\mathfrak{R}$ be an equivalence relation on Ω, $X \subset \Omega$ and $\tau_{\mathfrak{R}}(X) = \{ \Omega, \emptyset, \mathscr{L}_{\mathfrak{R}}(X), \mathbf{U}_{\mathfrak{R}}(X), \mathscr{B}_{\mathfrak{R}}(X)\}$ which satisfies the following axioms.

- $\Omega , \emptyset \in \tau_{\mathfrak{R}}(X)$.
- The union of elements of any subcollection of $\tau_{\mathfrak{R}}(X)$ is in $\tau_{\mathfrak{R}}(X)$.
- The intersection of any finite subcollection of elements of $\tau_{\mathfrak{R}}(X)$ is in $\tau_{\mathfrak{R}}(X)$.

Then $(\Omega, \tau_{\mathfrak{R}}(X))$ is called *nano topological space*. The members of $\tau_{\mathfrak{R}}(X)$ are called *nano open sets*.

III. MICRO TOPOLOGICAL SPACES

Definition: 4 [5] Let $(\Omega, \tau_{\Re}(X))$ be a nano topological space and *micro topology* is defined by

$$\mu_{\Re}(X) = \{K \cup (K' \cap S) : K, K' \in \tau_{\Re}(X), \text{ fixed } S \notin \tau_{\Re}(X)\}.$$

Definition: 5 [5] *The Micro topology* $\mu_{\Re}(X)$ satisfies the following postulates :

- $\Omega, \varnothing \in \mu_{\Re}(X)$.
- The union of elements of any subcollection of $\mu_{\Re}(X)$ is in $\mu_{\Re}(X)$.
- The intersection of any finite subcollection of elements of $\mu_{\Re}(X)$ is in $\mu_{\Re}(X)$.

Then $(\Omega, \tau_{\Re}(X), \mu_{\Re}(X))$ is called *Micro topological spaces* and the members of $\mu_{\Re}(X)$ are called *Micro open sets (μ-open sets)* and the complement of a μ-open set is called a μ-closed set.

IV. KASAJ TOPOLOGICAL SPACE

Definition: 6 [8] Let $(\Omega, \tau_{\Re}(X))$ be a nano topological space. Then *Kasaj topology* is defined by

$$KS_{\Re}(X) = \{(K \cap S) \cup (K' \cap S') : K, K' \in \tau_{\Re}(X), \text{ fixed } S, S' \notin \tau_{\Re}(X), S \cup S' = \Omega\}.$$

Definition: 7[8] The *Kasaj topology* $KS_{\Re}(X)$ satisfies the following postulates :

- $\Omega, \varnothing \in KS_{\Re}(X)$.
- The union of elements of any subcollection of $KS_{\Re}(X)$ is in $KS_{\Re}(X)$.
- The intersection of any finite subcollection of elements of $KS_{\Re}(X)$ is in $KS_{\Re}(X)$.

Then $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ is called *Kasaj topological space* and the members of $KS_{\Re}(X)$ are called *Kasaj open sets (KS-open sets)* and the complement of a *Kasaj-open set* is called a *Kasaj-closed (KS-closed) set* and the collection of all Kasaj-closed sets is denoted by KSCL(X).

Definition: 8[8] The Kasaj closure and the Kasaj interior of a set **P** is denoted by $KS_{cl}(P)$ and $KS_{int}(P)$, respectively. It is defined by $KS_{cl}(P) = \cap\{Q : P \subset Q, Q \text{ is KS-closed}\}$ and $KS_{int}(P) = \cup\{Q : Q \subset P, Q \text{ is KS-open}\}$.

Remark: 9[8]

- $KS_{int}(P)$ is the largest KS-open set contained in P.
- $KS_{cl}(P)$ is the smallest KS-closed set containing P.

Definition:10[8] For any two subsets P, Q of Ω in a *Kasaj topological space* $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$,

- P is a Kasaj-closed set if and only if $KS_{cl}(P) = P$.
- P is a Kasaj-open set if and only if $KS_{int}(P) = P$.
- If $P \subset Q$, then $KS_{int}(P) \subset KS_{int}(Q)$ and $KS_{cl}(P) \subset KS_{cl}(Q)$.
- $KS_{cl}(KS_{cl}(P)) = KS_{cl}(P)$ and $KS_{int}(KS_{int}(P)) = KS_{int}(P)$.
- $KS_{cl}(P) \cup KS_{cl}(Q) \subset KS_{cl}(P \cup Q)$.
- $KS_{int}(P) \cup KS_{int}(Q) \subset KS_{int}(P \cup Q)$.
- $KS_{cl}(P \cap Q) \subset KS_{cl}(P) \cap KS_{cl}(Q)$.
- $KS_{int}(P \cap Q) \subset KS_{int}(P) \cap KS_{int}(Q)$.
- $KS_{cl}(P) = [KS_{int}(P)]^c$.
- $KS_{int}(P) = [KS_{cl}(P)]^c$.

Definition: 11[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a *Kasaj topological space* and $P \subset \Omega$. Then P is called *Kasaj-pre-open(KS-pre-open) set* if $P \subset KS_{int}(KS_{cl}(P))$ and *Kasaj-pre-closed(KS-pre-closed) set* if $KS_{cl}(KS_{int}(P)) \subset P$. The set of all *Kasaj-pre-open sets* is denoted by $KSPO(\Omega, X)$.

Definition: 12[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a *Kasaj topological space* and $P \subset \Omega$. Then P is called *Kasaj-semi-open(KS-semi-open) set* if $P \subset KS_{cl}(KS_{int}(P))$ and *Kasaj-semi-closed(KS-semi-closed) set* if $KS_{int}(KS_{cl}(P)) \subset P$. The set of all *Kasaj-semi-open sets* is denoted by $KSSO(\Omega, X)$.

Definition: 13[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a *Kasaj topological space* and $P \subset \Omega$. Then P is called *Kasaj-α-open(KS-α-open) set* if $P \subset KS_{int}(KS_{cl}(KS_{int}(P)))$ and *Kasaj-α-closed(KS-α-closed) set* if $KS_{cl}(KS_{int}(KS_{cl}(P))) \subset P$. The set of all *Kasaj-α-open sets* is denoted by $KS\alpha O(\Omega, X)$.

Definition: 14[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a *Kasaj topological space* and $P \subset \Omega$. Then P is called *Kasaj-β-open(KS-β-open) set* if $P \subset KS_{cl}(KS_{int}(KS_{cl}(P)))$ and *Kasaj-β-closed(KS-β-closed) set* if $KS_{int}(KS_{cl}(KS_{int}(P))) \subset P$. The set of all *Kasaj-β-open sets* is denoted by $KS\beta O(\Omega, X)$.

Example: 15 Let $\Omega = \{\lambda, \theta, \pi, \upsilon, \kappa\}$ with $\Omega/\Re = \{\{\lambda\}, \{\theta, \pi\}, \{\upsilon, \kappa\}\}$ and $X = \{\lambda, \pi\} \subset \Omega$. Then $\tau_{\Re}(X) = \{\varnothing, \Omega, \{\lambda\}, \{\theta,\pi\}, \{\lambda, \theta, \pi\}\}$. If we consider $S = \{\lambda, \theta, \upsilon\}$ and $S' = \{\pi, \kappa\}$, then

- $KS_{\Re}(X) = \{\varnothing, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \Omega\}$.
- $KSPO(\Omega, X) = \{\varnothing, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \Omega\}$.

- $KSSO(\Omega, X) = \{\emptyset, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\lambda, \upsilon\}, \{\theta, \upsilon\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\lambda, \pi, \upsilon\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon, \kappa\}, \Omega\}$.
- $KS\alpha O(\Omega, X) = \{\emptyset, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \Omega\}$.
- $KS\beta O(\Omega, X) = \{\emptyset, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\lambda, \upsilon\}, \{\theta, \upsilon\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\lambda, \pi, \upsilon\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon, \kappa\}, \Omega\}$.

Example: 16 Let $\Omega = \{\lambda, \theta, \pi, \upsilon, \kappa\}$ with $\Omega/\Re = \{\{\lambda\}, \{\theta, \pi, \upsilon\}, \{\kappa\}\}$ and $X = \{\upsilon, \kappa\} \subset \Omega$. Then $\tau_{\Re}(X) = \{\emptyset, \Omega, \{\kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}\}$. If we consider $S = \{\lambda, \theta, \kappa\}$ and $S' = \{\pi, \upsilon\}$, then

- $KS_{\Re}(X) = \{\emptyset, \{\theta\}, \{\kappa\}, \{\theta, \kappa\}, \{\pi, \upsilon\}, \{\lambda, \theta, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \Omega\}$.
- $KSPO(\Omega, X) = \{\emptyset, \{\theta\}, \{\pi\}, \{\upsilon\}, \{\kappa\}, \{\theta, \pi\}, \{\theta, \upsilon\}, \{\upsilon, \kappa\}, \{\pi, \upsilon\}, \{\lambda, \theta, \kappa\}, \{\theta, \pi, \kappa\}, \{\theta, \upsilon, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \theta, \upsilon, \kappa\}, \Omega\}$.
- $KSSO(\Omega, X) = \{\emptyset, \{\theta\}, \{\kappa\}, \{\lambda, \theta\}, \{\lambda, \kappa\}, \{\theta, \kappa\}, \{\pi, \upsilon\}, \{\lambda, \theta, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \pi, \upsilon, \kappa\}, \Omega\}$.
- $KS\alpha O(\Omega, X) = \{\emptyset, \{\theta\}, \{\kappa\}, \{\theta, \kappa\}, \{\pi, \upsilon\}, \{\lambda, \theta, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \Omega\}$.
- $KS\beta O(\Omega, X) = \{\emptyset, \{\theta\}, \{\pi\}, \{\upsilon\}, \{\kappa\}, \{\lambda, \theta\}, \{\lambda, \kappa\}, \{\theta, \pi\}, \{\theta, \kappa\}, \{\pi, \kappa\}, \{\upsilon, \kappa\}, \{\pi, \upsilon\}, \{\theta, \kappa\}, \{\lambda, \theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \kappa\}, \{\lambda, \upsilon, \kappa\}, \{\theta, \pi, \kappa\}, \{\theta, \upsilon, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \theta, \upsilon, \kappa\}, \{\lambda, \pi, \upsilon, \kappa\}, \Omega\}$.

V. KASAJ-B-OPEN SETS

In this section, we define and study the forms of Kasaj-b-open (KS-b-open) sets.

Definition: 17 Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a Kasaj topological space and $P \subset \Omega$. Then P is said to be Kasaj-b-open (KS-b-open) set if $P \subset KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))$. The complement of KS-b-open set is called KS-b-closed set. The set of all KS-b-open set is denoted by $KSbO(\Omega, X)$.

Remark: 18

- In Example 15, we have $KSbO(\Omega, X) = \{\emptyset, \{\lambda\}, \{\theta\}, \{\pi\}, \{\pi, \kappa\}, \{\lambda, \pi\}, \{\lambda, \theta\}, \{\lambda, \upsilon\}, \{\theta, \upsilon\}, \{\theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \theta, \pi\}, \{\lambda, \pi, \upsilon\}, \{\lambda, \pi, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon, \kappa\}, \Omega\}$.
- In Example 16, we have $KSbO(\Omega, X) = \{\emptyset, \{\theta\}, \{\pi\}, \{\upsilon\}, \{\kappa\}, \{\lambda, \theta\}, \{\lambda, \kappa\}, \{\theta, \pi\}, \{\theta, \upsilon\}, \{\pi, \kappa\}, \{\upsilon, \kappa\}, \{\pi, \upsilon\}, \{\theta, \kappa\}, \{\lambda, \theta, \pi\}, \{\lambda, \theta, \upsilon\}, \{\lambda, \pi, \kappa\}, \{\lambda, \theta, \kappa\}, \{\lambda, \upsilon, \kappa\}, \{\theta, \pi, \kappa\}, \{\theta, \upsilon, \kappa\}, \{\pi, \upsilon, \kappa\}, \{\theta, \pi, \upsilon\}, \{\theta, \pi, \upsilon, \kappa\}, \{\lambda, \theta, \pi, \upsilon\}, \{\lambda, \theta, \pi, \kappa\}, \{\lambda, \theta, \upsilon, \kappa\}, \{\lambda, \pi, \upsilon, \kappa\}, \Omega\}$.

Theorem: 19 $\mathbf{KS}_{\Re}(X) \subset KSbO(\Omega, X)$.

Proof: Let $P \in \mathbf{KS}_{\Re}(X)$, i.e., $P = KS_{int}(P)$. Since $P \subset KS_{cl}(P)$ for all $P \subset \Omega$, therefore,

$$P = KS_{int}(P) \subset KS_{int}(KS_{cl}(P))$$

and

$$P = KS_{int}(P) \subset KS_{cl}(KS_{int}(P))$$

which implies that

$$P = KS_{int}(P) \subset KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P)).$$

Hence $P \in KSbO(\Omega, X)$.

Remark: 20 In general, $KSbO(\Omega, X) \not\subset \mathbf{KS}_{\Re}(X)$. In Example 16, one can see that $\{\pi\} \in KSbO(\Omega, X)$ but $\{\pi\} \notin KS_{\Re}(X)$.

Theorem: 21

I. $KSPO(\Omega, X) \subset KSbO(\Omega, X)$.
II. $KSSO(\Omega, X) \subset KSbO(\Omega, X)$.

Proof(I): Let $P \in KSPO(\Omega, X)$ (i.e., $P \subset KS_{int}(KS_{cl}(P))$) which implies that $P \subset KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))$. Hence $P \in KSbO(\Omega, X)$.

Proof(II): Let $P \in KSSO(\Omega, X)$ (i.e., $P \subset KS_{cl}(KS_{int}(P))$) which implies that $P \subset KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))$. Hence $P \in KSbO(\Omega, X)$.

Remark: 22 In general, $KSbO(\Omega, X) \not\subset KSPO(\Omega, X)$. In Example 16, one can see that $\{\lambda, \theta, \pi\} \in KSbO(\Omega, X)$ but $\{\lambda, \theta, \pi\} \notin KSPO(\Omega, X)$ and $\{\lambda, \theta, \pi\} \notin KSSO(\Omega, X)$.

Theorem: 23 $KS\alpha O(\Omega, X) \subset KSbO(\Omega, X)$.
Proof: By Theorem 3.6 of **[10]** and Theorem 21, the proof follows.

Theorem: 24 $KSbO(\Omega, X) \subset KS\beta O(\Omega, X)$.

Proof: Let $P \in KSbO(\Omega, X)$, i.e., $P \subseteq KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))$. Since $P \subseteq KS_{cl}(P)$ for all $P \subseteq \Omega$ and by Definition 10, we observe that

$$KS_{int}(KS_{cl}(P)) \subseteq KS_{cl}(KS_{int}(KS_{cl}(P)))$$

and

$$KS_{cl}(KS_{int}(P)) \subseteq KS_{cl}(KS_{int}(KS_{cl}(P)))$$

which implies that

$$P \subseteq KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P)) \subseteq KS_{cl}(KS_{int}(KS_{cl}(P))).$$

Hence, $P \in KS\beta O(\Omega, X)$.

Remark: 25 In general, $KS\beta O(\Omega, X) \not\subset KSbO(\Omega, X)$.

Definition: 26 The Kasaj-b-closure and Kasaj-b-interior of a set P are denoted by KS-b_{cl}(P) and KS-b_{int}(P), respectively and are defined by

- KS-$b_{cl}(P) = \cap\{Q : P \subseteq Q$ and $Q \in KSbCL(\Omega, X)\}$ and
- KS-$b_{int}(P) = \cup\{Q : Q \subseteq P$ and $Q \in KSbO(\Omega, X)\}$.

Remark: 27 It is clear that KS-b_{cl}(P) is the smallest closed set containing P and KS-b_{int}(P) is the largest open set contained in P.

Theorem: 28

I. P = KS-b_{cl}(P) iff $P \in KSbCL(\Omega, X)$.

II. P = KS-b_{int}(P) iff $P \in KSbO(\Omega, X)$.

Proof(I) : ($\Rightarrow$)Let $P \in KSbCL(\Omega, X)$. Then $P \subseteq$ KS-b_{cl}(P). and KS-$b_{cl}(P) = \cap\{Q : P \subseteq Q, Q \in KSbCL(\Omega, X)\}$. Since $P \in KSbCL(\Omega, X)$, P is an element of $\{Q : P \subseteq Q, Q \in KSbCL(\Omega, X)\}$. So, $\cap\{Q : P \subseteq Q, Q \in KSbCL(\Omega, X)\} = P$. Hence P = KS-$b_{cl}$(P).

($\Leftarrow$) Let P = KS-b_{cl}(P). Then by Remark 27, $P \in KSbCL(\Omega, X)$.

Proof(II) : ($\Rightarrow$) Let $P \in KSbO(\Omega, X)$. Then $P \subseteq \cup\{Q : Q \subseteq P$ and $Q \in KSbO(\Omega, X)\}$ = KS-b_{int}(P). So, $P \subseteq$ KS-b_{int}(P). As $P \in KSbO(\Omega, X)$, KS-$b_{int}(P) \subseteq P$. Hence P = KS-b_{int}(P).

($\Leftarrow$) Assume that P = KS-b_{int}(P). Then by Remark 27, $P \in KSbO(\Omega, X)$.
Hence, we get desired.

Proposition: 28 If $P \in KS\alpha O(\Omega, X)$ and $Q \in KSbO(\Omega, X)$, then $P \cup Q \in KSbO(\Omega, X)$.

Proof: Assume that $P \in KS\alpha O(\Omega, X)$ and $Q \in KSbO(\Omega, X)$, i.e., $P \subseteq KS_{int}(KS_{cl}(KS_{int}(P)))$ and $Q \subseteq KS_{int}(KS_{cl}(Q)) \cup KS_{cl}(KS_{int}(Q))$. By Theorem 21, $P \in KSbO(\Omega, X)$. i.e., $P \subseteq KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))$.
Now,

$$\begin{aligned} P \cup Q &\subseteq (KS_{int}(KS_{cl}(P)) \cup KS_{cl}(KS_{int}(P))) \cup (KS_{int}(KS_{cl}(Q)) \cup KS_{cl}(KS_{int}(Q))) \\ &= (KS_{int}(KS_{cl}(P)) \cup (KS_{int}(KS_{cl}(Q))) \cup (KS_{cl}(KS_{int}(P))) \cup KS_{cl}(KS_{int}(Q))) \\ &\subseteq (KS_{int}(KS_{cl}(P) \cup KS_{cl}(Q))) \cup (KS_{cl}(KS_{int}(P) \cup KS_{int}(Q))) \\ &\subseteq (KS_{int}(KS_{cl}(P \cup Q))) \cup (KS_{cl}(KS_{int}(P \cup Q))) \end{aligned}$$

Hence $P \cup Q \in KSbO(\Omega, X)$.

Theorem: 29 $\cup_{\alpha\in\Lambda} P_\alpha \in KSbO(\Omega, X)$ whenever $P_\alpha \in KSbO(\Omega, X)$ and Λ is an index set.

Proof: Let $\{P_\alpha : \alpha \in \Lambda\} \in KSbO(\Omega, X)$. i.e., for each α, $P_\alpha \subseteq KS_{int}(KS_{cl}(P_\alpha)) \cup KS_{cl}(KS_{int}(P_\alpha))$. Now,

$$\begin{aligned} \cup_{\alpha\in\Lambda} P_\alpha &\subseteq \cup_{\alpha\in\Lambda} \{KS_{int}(KS_{cl}(P_\alpha)) \cup KS_{cl}(KS_{int}(P_\alpha))\} \\ &\subseteq \{\cup_{\alpha\in\Lambda} KS_{int}(KS_{cl}(P_\alpha))\} \cup \{\cup_{\alpha\in\Lambda} KS_{cl}(KS_{int}(P_\alpha))\} \\ &\subseteq \{KS_{int}(\cup_{\alpha\in\Lambda} KS_{cl}(P_\alpha))\} \cup \{KS_{cl}(\cup_{\alpha\in\Lambda} KS_{int}(P_\alpha))\} \\ &\subseteq \{KS_{int}(KS_{cl}(\cup_{\alpha\in\Lambda} P_\alpha))\} \cup \{KS_{cl}(KS_{int}(\cup_{\alpha\in\Lambda} P_\alpha))\} \end{aligned}$$

Hence, $\cup_{\alpha\in\Lambda} P_\alpha \in KSbO(\Omega, X)$.

Theorem: 30 $\cap_{\alpha\in\Lambda} P_\alpha \in KSbCL(\Omega, X)$ whenever $P_\alpha \in KSbCL(\Omega, X)$ and Λ is an index set.

Proof: Let $\{P_\alpha : \alpha \in \Lambda\} \in KSbCL(\Omega, X)$. i.e., for each α, $(P_\alpha)^c \in KSbO(\Omega, X)$. By Theorem 29, $\cup_{\alpha\in\Lambda} (P_\alpha)^c \in KSbO(\Omega, X)$ which implies that $\cap_{\alpha\in\Lambda} P_\alpha = (\cup_{\alpha\in\Lambda} (P_\alpha)^c)^c$. So, we get desired.

Theorem: 31

I. KS-$b_{int}(P \cup Q) \supseteq$ KS-b_{int}(P) $\cup$ KS-b_{int}(Q).

II. KS-$b_{int}(P \cap Q)$ = KS-b_{int}(P) $\cap$ KS-b_{int}(Q).

Proof(I): We know that KS-$b_{int}(P) \subseteq P$ and KS-$b_{int}(Q) \subseteq Q$ implies KS-b_{int}(P) $\cup$ KS-$b_{int}(Q) \subseteq P \cup Q$. Since KS-$b_{int}$(P), KS-$b_{int}(Q) \in KSbO(\Omega, X)$. Hence by Theorem 29, KS-b_{int}(P) $\cup$ KS-$b_{int}(Q) \in KSbO(\Omega, X)$. and KS-$b_{int}$(P) $\cup$ KS-$b_{int}(Q) \subseteq P \cup Q$. Since KS-$b_{int}(P \cup Q)$ is the largest KS-b-open set contained in $P \cup Q$. It follows that KS-$b_{int}(P \cup Q) \supseteq$ KS-b_{int}(P) $\cup$ KS-b_{int}(Q).

Proof(II) : Let $x \in KS\text{-}b_{int}(P \cap Q)$. Then there is a KS-b-open set D such that $x \in D \subseteq P \cap Q$. i.e., there exists a KS-b-open set such that $x \in D \subseteq P$ and $x \in D \subseteq Q$. Hence $x \in KS\text{-}b_{int}(P)$ and $x \in KS\text{-}b_{int}(Q)$ i.e., $x \in KS\text{-}b_{int}(P) \cap KS\text{-}b_{int}(Q)$. Thus, $KS\text{-}b_{int}(P \cap Q) \subseteq KS\text{-}b_{int}(P) \cap KS\text{-}b_{int}(Q)$. Similarly, we can prove reverse inclusion.

VI. Kasaj-regular-open Set

In this section, we define and study the forms of Kasaj-regular-open (KS-regular-open) sets.

Definition: 32
Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ be a Kasaj topological space and $P \subseteq \Omega$. Then P is said to be Kasaj-regular-open (briefly, KS-regular-open) set if $P = KS_{int}(KS_{cl}(P))$. The complement of KS-regular-open set is called KS-regular-closed set. The set of all KS-regular-open set is denoted by $KSRO(\Omega, X)$.

Remark: 33

- In Example 15, we have $KSRO(\Omega, X) = \{ \emptyset, \{\lambda\}, \{\theta\}, \{\pi, \kappa\}, \{\theta, \pi, \kappa\}, \{\lambda, \pi, \kappa\}, \Omega\}$.
- In Example 16, we have $KSRO(\Omega, X) = \{\emptyset, \{\theta\}, \{\kappa\}, \{\pi, \upsilon\}, \{ \lambda, \theta, \kappa\}, \{\pi, \upsilon, \kappa\} , \{\theta, \pi, \upsilon\} , \Omega\}$.

Remark: 34 From these examples one can observe that $KSRO(\Omega, X)$ is not closed under union.

Theorem: 35 $KSRO(\Omega, X) \subseteq KS_{\Re}(X)$.

Proof: Let $P \in KSRO(\Omega, X)$, i.e., $P = KS_{int}(KS_{cl}(P))$. Then $KS_{int}(P) = KS_{int}(KS_{int}(KS_{cl}(P))) = KS_{int}(KS_{cl}(P))$. Hence $P \in KS_{\Re}(X)$, i.e., P is KS-open set..

Remark: 36 In general, $KS_{\Re}(X) \not\subset KSRO(\Omega, X)$. In Example 16, $\{\theta, \kappa\} \in KS_{\Re}(X)$ but $\{\theta, \kappa\} \notin KSRO(\Omega, X)$.

VII. Kasaj-B-continuous functions

Definition: 37[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-continuous (KS-continuous) function if $f^{-1}(D) \in KS_{\Re}(X)$ whenever $D \in KS_{\Re'}(Y)$.

Theorem: 38[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is KS-continuous function if and only if $f^{-1}(D) \in KSCL(X)$ whenever $D \in KSCL(Y)$.

Definition: 39[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-pre-continuous function if $f^{-1}(D) \in KSPCL(\Omega, X)$ whenever $D \in KSCL(Y)$.

Definition: 40[8] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-semi-continuous function if $f^{-1}(D) \in KSSCL(\Omega, X)$ whenever $D \in KSCL(Y)$.

Definition: 41[11] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-α-continuous function if $f^{-1}(D) \in KS\alpha CL(\Omega, X)$ whenever $D \in KSCL(Y)$.

Definition: 42[11] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-β-continuous function if $f^{-1}(D) \in KS\beta CL(\Omega, X)$ whenever $D \in KSCL(Y)$.

Now, we define KS-b-continuous function.

Definition: 43[11] Let $(\Omega, \tau_{\Re}(X), KS_{\Re}(X))$ and $(\Omega', \tau_{\Re'}(Y), KS_{\Re'}(Y))$ be two Kasaj topological spaces, $X \subseteq \Omega$ and $Y \subseteq \Omega'$. Then $f : \Omega \to \Omega'$ is Kasaj-b-continuous function if $f^{-1}(D) \in KSbCL(\Omega, X)$ whenever $D \in KSCL(Y)$.

Theorem: 44 Every KS-continuous function is KS-b-continuous function.

Proof: Let $f : \Omega \to \Omega'$ be a KS-continuous function, i.e., $f^{-1}(D) \in KSCL(X)$ whenever $D \in KSCL(Y)$. By Theorem 19, Therefore $f^{-1}(D) \in KSbCL(\Omega, X)$ for all $D \in KSCL(Y)$. Hence, f is KS-b-continuous function.

Theorem: 45 Every KS-b-continuous function is KS-β-continuous.

Proof: Let $f : \Omega \to \Omega'$ be a KS-b-continuous function, i.e., $f^{-1}(D) \in KSbCL(\Omega, X)$ whenever $D \in KSCL(Y)$. By Theorem 24, Therefore $f^{-1}(D) \in KS\beta CL(\Omega, X)$ for all $D \in KSCL(Y)$. Hence, f is KS-β-continuous function.

Theorem: 46 Every KS-α-continuous function is KS-b-continuous.

Proof: Let $f : \Omega \to \Omega'$ be a KS-α-continuous function, i.e., $f^{-1}(D) \in KS\alpha CL(\Omega, X)$ whenever $D \in KSCL(Y)$. By Theorem 23, Therefore $f^{-1}(D) \in KSbCL(\Omega, X)$ for all $D \in KSCL(Y)$. Hence, f is KS-b-continuous function.

Theorem: 47 Every KS-pre-continuous function is KS-b-continuous.

Proof: Let $f : \Omega \to \Omega'$ be a KS-pre-continuous function, i.e., $f^{-1}(D) \in KSPCL(\Omega, X)$ whenever $D \in KSCL(Y)$. By Theorem 21, Therefore $f^{-1}(D) \in KSbCL(\Omega, X)$ for all $D \in KSCL(Y)$. Hence, f is KS-b-continuous function.

Theorem: 48 Every KS-semi-continuous function is KS-b-continuous.

Proof: Let $f : \Omega \to \Omega'$ be a KS-semi-continuous function, i.e., $f^{-1}(D) \in KSSCL(\Omega, X)$ whenever $D \in KSCL(Y)$. By Theorem 21, Therefore $f^{-1}(D) \in KSbCL(\Omega, X)$ for all $D \in KSCL(Y)$. Hence, f is KS-b-continuous function.

VIII. Conclusion

From theorems of Section V, VI and theorems of the paper **[8]** and **[11]** we have the following implications.

$KSRO(\Omega, X) \subseteq KS_{\mathfrak{R}}(X) \subseteq KS\alpha O(\Omega, X) \subseteq KSbO(\Omega, X) \subseteq KS\beta O(\Omega, X)$,
$KS\alpha O(\Omega, X) \subseteq KSPO(\Omega, X) \subseteq KSbO(\Omega, X)$ &
$KS\alpha O(\Omega, X) \subseteq KSSO(\Omega, X) \subseteq KSbO(\Omega, X)$

IX. Acknowledgment

The first author gratefully acknowledges a Junior Research Fellowship from University Grant Commission (UGC), India.

References

[1] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On pre-continuous mappings and weak pre-continuous mappings, Proc. Math. and Phys. Soc. Egypt, 53, 47-53.
[2] D. A. Mary and A. Arokia rani, On semi pre-open sets in nano topological spaces, Math. Sci. Int. Res. Jnl, 3, 771-773.
[3] M. L. Thivagar and C. Richard, On Nano Forms of weakly open sets, International Journal of Mathematics and Statistic invention, 1/1, 31-37.
[4] N. Levine, Semi open sets and semi continuity in Topological spaces, Amer. Math. Monthly, 70, 36-41.
[5] S. Chandrasekar, On Micro Topological Spaces, Journal of New Theory, 26, 23-31.
[6] M. Caldas, A note on some applications of α-open sets}, Int. Jour. Mat. and Mathl.sci., 2, 125-130.
[7] M. E. Abd El-monsef, S. N. El-Dech and R. A. Mahmound, β-open and β-continuous mappings, Bull. Fac. Sci. Assiut Univ., 12, 77-90.
[8] K. G. Rachchh and S. I. Ghanchi, On Kasaj Topological Spaces, Mal. Jour. Mat. 8(4), 1766-1770.
[9]D. Andrijevic, b-open sets, Math. Vesnik, 48(1), 59-64.
[10] K. F. Porter, Regular open sets in topology, Int. Jour. of Math. and math.sci., 2, 299-302.
[11] K. G. Rachchh and S. I. Ghanchi, On Kasaj-alpha-open sets and Kasaj-beta-open sets in Kasaj Topological Spaces, Communicated