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On total domination in some path related graphs

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Abstract

If G is a graph with vertex set $V(G)$ then dominating set $D \subseteq V(G)$ is called total if every vertex of $V(G)$ is adjacent to at least one vertex of D while it is called equitable if for every vertex u in $V(G) - D$ there exists a vertex v in D such that the degree difference between these vertices is at most one and uv is an edge in G. A dominating set which is both total and equitable is called total equitable dominating set. The minimum cardinality of a total dominating set of G is called the total domination number of G which is denoted by $\gamma_t(G)$. The total equitable domination number of G is the minimum cardinality of total equitable dominating set of G and is denoted by $\gamma_t^e(G)$. We determine the exact values of total domination number as well as total equitable domination number of some path related graphs.

Keywords: dominating set, equitable dominating set, total dominating set
AMS Subject Classification(2010): 05C69

1 Introduction

We begin with a finite, simple, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the open neighborhood $N(v)$ of v is defined as $N(v) = \{u \in V(G) : uv \in E(G)\}$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n.

The set $D \subseteq V(G)$ of vertices in a graph G is called dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$. Many domination models are introduced in recent past. Global domination, total domination, independent domination, equitable domination are some among them worth to be mentioned.

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Motivated by the concepts of total domination and equitable domination, a new concept of total equitable domination was conceived by Basavanagoud *et al.* [1] and further explored by Vaidya and Parmar [7]. In the present paper, we obtain the total domination number and total equitable domination number of some path related graphs.

A subset D of $V(G)$ is called an equitable dominating set if for every $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(v) - d(u)| \leqslant 1$, where $d(u)$ denotes the degree of vertex u and $d(v)$ denotes the degree of vertex v. The minimum cardinality of such dominating set is called the equitable domination number of G which is denoted by $\gamma^e(G)$. This concept was introduced by Swaminathan and Dharmalingam [6]. A subset D of $V(G)$ is called a total dominating set of G if $N(D) = V(G)$ or equivalently if every vertex $v \in V(G)$ is adjacent to at least one element in D. The minimum cardinality of total dominating set is called total domination number which is denoted by $\gamma_t(G)$. This concept was introduced by Cockayne *et al.* [2]. A dominating set which is both total and equitable is called total equitable dominating set. The total equitable domination number of G is the minimum cardinality of a total equitable dominating set of G which is denoted by $\gamma_t^e(G)$. This concept was introduced by Basavanagoud *et al.* [1].

For standard terminology and notation in graph theory we rely upon West [8] while for any undefined term related to theory of domination we refer to Haynes *et al.*[4].

2 Main Results

Definition 2.1. The square of a graph G denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance of 1 or 2 apart in G.

Theorem 2.2. If $G = P_n^2$ then

$$\gamma_{\mathrm{t}}(G) = \begin{cases} 2\left\lfloor \frac{n}{7} \right\rfloor + 1 & ; \text{ if } n \equiv 1 \text{ or } 2 \pmod 7 \\ \\ 2\left\lceil \frac{n}{7} \right\rceil & ; \text{ if } n \not\equiv 1 \text{ or } 2 \pmod 7 \end{cases}$$

Proof: Let $V(P_n) = \{v_1, v_2, v_3, ..., v_n\} = V(G)$ be the vertex set where $d_G(v_1) = d_G(v_n) = 2$, $d_G(v_2) = d_G(v_{n-1}) = 3$ and $d_G(v_i) = 4$, for all $i \in \{3, 4, 5, ..., n-2\}$.

If D is any total dominating set of G then it is obvious that v_3 must belong to D as $d_G(v_3) = 4 = \Delta(G)$.

To prove the result we consider the following cases:

Case 1: $n \equiv 1$ or $2 (\mathrm{mod} 7)$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \left\lfloor \frac{n}{7} \right\rfloor - 1\} \cup \{v_{n-1}\} & ; \text{ if } n \equiv 1 \pmod 7 \\ \\ \{v_{7i+3}, v_{7i+5} / 0 \leq i \leq \left\lfloor \frac{n}{7} \right\rfloor - 1\} \cup \{v_{n-2}\} & ; \text{ if } n \equiv 2 \pmod 7 \end{cases}$$

Then $|D| = 2\left\lfloor \frac{n}{7} \right\rfloor + 1$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex. Further D is a minimal total dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. We claim that $|D|$ is minimum because any $v \in D$ will dominate maximum number of distinct vertices of G as $d_G(v) = 4 = \Delta(G)$. Therefore, any set containing the vertices less than that of $|D|$ cannot be a total dominating set of G.

Hence, $\gamma_t(G) = 2\left\lfloor \frac{n}{7} \right\rfloor + 1$, when $n \equiv 1$ or $2(\mathrm{mod}7)$.

Case 2: $n \not\equiv 1$ or $2(\mathrm{mod}7)$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+3}, v_{7i+5}/0 \le i \le \lfloor \frac{n}{7} \rfloor\} & ;\ \text{if } n \equiv 0 \text{ or } 5 \text{ or } 6\,(\text{mod } 7) \\ \{v_{7i+3}, v_{7i+5}/0 \le i \le \lfloor \frac{n}{7} \rfloor - 1\} \cup \{v_{n-1}, v_n\} & ;\ \text{if } n \equiv 3 \text{ or } 4\,(\text{mod } 7) \end{cases}$$

Then $|D| = 2\left\lceil \frac{n}{7} \right\rceil$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex. Further D is a minimal total dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. We claim that $|D|$ is minimum because any $v \in D$ will dominate maximum number of distinct vertices of G as $d_G(v) = 4 = \Delta(G)$. Therefore, any set containing the vertices less than that of $|D|$ cannot be a total dominating set of G.

Hence, $\gamma_t(G) = 2\left\lceil \frac{n}{7} \right\rceil$, when $n \not\equiv 1$ or $2(\mathrm{mod}7)$. ■

Illustration 2.3. In Figure 1, the graph P_7^2 is shown in which the set of solid vertices is its total dominating set of minimum cardinality. It is the case $n \not\equiv$ 1 or 2(mod 7).

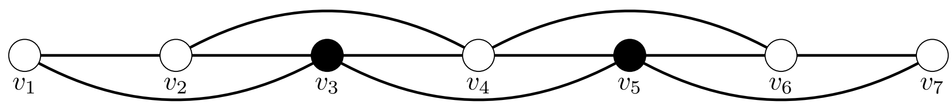

Figure 1: $\gamma_t(P_7^2) = 2$

Theorem 2.4. If $G = P_n^2$ then

$$\gamma_t^e(G) = \begin{cases} 2\lceil \frac{n}{7} \rceil + 1 & ;\ \text{if } n \equiv 0 \text{ or } 6\,(\text{mod } 7) \\ 2\lceil \frac{n}{7} \rceil & ;\ \text{otherwise} \end{cases}$$

Proof: If D is any total equitable dominating set of G then it is obvious that v_2 must belong to D as $|d_G(v_1) - d_G(v_2)| = 1$ and $|d_G(v_2) - d_G(v_3)| = 1$.

To prove this result we consider the following cases:

Case 1: $n \equiv 0$ or $6 (\text{mod } 7)$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lceil \frac{n}{7} \rceil - 1\} \cup \{v_{n-1}\} & ; \text{ if } n \equiv 0 \,(\text{mod } 7) \\ \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lceil \frac{n}{7} \rceil - 1\} \cup \{v_{n}\} & ; \text{ if } n \equiv 6 \,(\text{mod } 7) \end{cases}$$

Then $|D| = 2 \lceil \frac{n}{7} \rceil + 1$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex and D is also an equitable dominating set of G as for any $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimal total equitable dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. We claim that $|D|$ is minimum because any $v \in D$ will dominate maximum four vertices as $d_G(v_2) = 3$ and $d_G(v_i) = 4 = \Delta(G)$ for all $3 \leq i \leq n - 2$.

Hence, $\gamma_t^e(G) = 2 \lceil \frac{n}{7} \rceil + 1$.

Case 2: $n \not\equiv 0$ or $6 (\text{mod} 7)$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor - 1\} \cup \{v_{n-2}, v_n\} & ; \text{ if } n \equiv 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \,(\text{mod } 7) \\ \{v_{7i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{7} \rfloor\} & ; \text{ if } n \equiv 5 \,(\text{mod } 7) \end{cases}$$

Then $|D| = 2 \lceil \frac{n}{7} \rceil$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex and D is also an equitable dominating set of G as for any vertex $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimal total equitable dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. We claim that $|D|$ is minimum because any $v \in D$ will dominate maximum four vertices as $d(v_2) = 3$ and $d_G(v_i) = 4 = \Delta(G)$ for all $3 \leq i \leq n - 2$.

Hence, $\gamma_t^e(G) = 2 \lceil \frac{n}{7} \rceil$. ■

Illustration 2.5. In Figure 2, the graph P_7^2 is shown in which the set of solid vertices is its total equitable dominating set with minimum cardinality. It is the case $n \equiv 0 \ (\text{mod } 7)$.

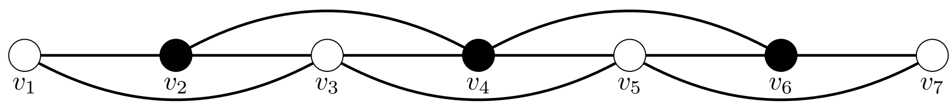

Figure 2: $\gamma_t^e(P_7^2) = 3$

Definition 2.6. Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup ... \cup S_t \cup T$, where each S_i is a set of all the vertices having same degree (at least 2 vertices) and $T = V(G) \setminus \bigcup_{i=1}^{t} S_i$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, ..., w_t$ and joining to each vertex of S_i for $1 \leqslant i \leqslant t$.

Theorem 2.7. If G is the graph obtained by degree splitting of P_n then

$$\gamma_t(G) = \begin{cases} n-1 & ; \text{ if } n = 3,4 \\ 4 & ; \text{ if } n > 4 \end{cases}$$

Proof: The path P_n have two pendant vertices and remaining $n-2$ vertices of degree two. Then $V(P_n) = \{v_i \ : \ 1 \le i \le n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i \ : \ 2 \le i \le n-1\}$. To obtain G from P_n add two vertices x and y corresponding to S_1 and S_2 respectively. Thus $V(G) = V(P_n) \cup \{x, y\}$. And $E(G) = E(P_n) \cup \{xv_i \ : \ v_i \in S_1\} \cup \{yv_j \ : \ v_j \in S_2\}$.

Suppose D is any total dominating set of G. Then it is obvious that y must belong to D as $\Delta(G) = n-2 = d_G(y)$.

For $n \le 4$, we construct a set of vertices D as follows:

$$D = \begin{cases} \{v_1, v_2\} & ; \text{ if } n = 3 \\ \{v_1, v_2, v_3\} & ; \text{ if } n = 4 \end{cases}$$

Then $|D| = n-1$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex. And D is obviously minimal total dominating set of G with minimum cardinality. Hence $\gamma_t(G) = n-1$, for $n \le 4$.

For $n > 4$, we construct a dominating set $D = \{y, v_1, v_2, v_{n-1}\}$ with $|D| = 4$. Also D is total dominating set of G as $\langle D \rangle$ has no isolated vertex. Further D is a minimal total dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. We claim that $|D|$ is minimum because $\Delta(G) = n-2 = d_G(y)$ will dominate maximum number of distinct vertices of G, $\{v_2, v_{n-1}\}$ dominate $\{v_1, v_n\}$ while v_1 and v_n dominate x. Thus, $\gamma_t(G) = 4$, for all $n > 4$. ■

Illustration 2.8. In Figure 3, the graph $DS(P_7)$ is shown in which the set of solid vertices is its total dominating set with minimum cardinality.

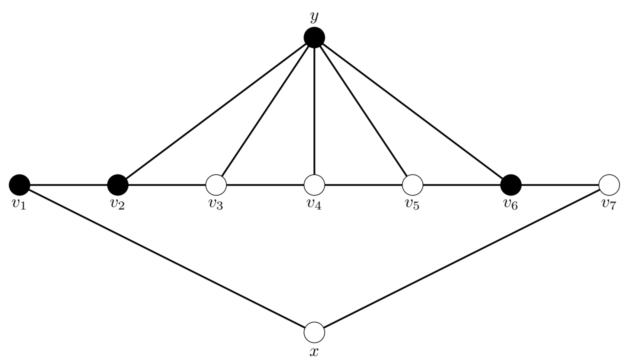

Figure 3: $\gamma_t(DS(P_7)) = 4$

Theorem 2.9. $\gamma_t^e(G) = \begin{cases} 3 & ; \text{ if } n = 4 \\ \left\lfloor \dfrac{n}{3} \right\rfloor + 2 & ; \text{ if } n \geq 5 \end{cases}$

Proof: If D is any total equitable dominating set of G then it is obvious that v_1 must belong to D as $|d_G(v_1) - d_G(v_2)| = 1$ and $|d_G(v_1) - d_G(x)| = 0$.

To prove the result we consider the following cases:

Case 1: $n = 4$

We construct a set of vertices as $D = \{v_1, v_2, v_3\}$.

Then $|D| = 3$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex and D is also equitable dominating set of G as every vertex $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G))$ and $|d_G(u) - d_G(v)| \leq 1$.

Further D is a minimal total equitable dominating set of G because for any $v \in D$, $\langle D - \{v\} \rangle$ has an isolated vertex. Therefore $|D|$ is minimum. Hence $\gamma_t^e(G) = 3$.

Case 2: $n \geq 5$

We construct a set of vertices D as follows:

$$D = \begin{cases} \{v_{3i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\} \cup \{y, v_{n-1}, v_n\} & ; \text{ if } n \equiv 0 \text{ or } 2 \pmod 3 \\ \{v_{3i+2}, v_{7i+4} / 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1\} \cup \{y, v_n\} & ; \text{ if } n \equiv 1 \pmod 3 \end{cases}$$

Then $|D| = \left\lceil \dfrac{n}{3} \right\rceil + 2$. Also D is a total dominating set of G as $\langle D \rangle$ has no isolated vertex and D is also an equitable dominating set of G as every vertex $v \in V(G) - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d_G(u) - d_G(v)| \leq 1$.

Further we assume that $D - \{v\}$ is a minimal total equitable dominating set of G for any $v \in D$. For a vertex v, $|d_G(u) - d_G(v)| \geq 1$. This is contradiction to $D - \{v\}$ is a total equitable dominating set of G. Thus D is a minimum total equitable dominating set of G. Hence the theorem is proved. ■

Illustration 2.10. In Figure 4, the graph $DS(P_7)$ is shown in which the set of solid vertices is its total equitable dominating set with minimum cardinality.

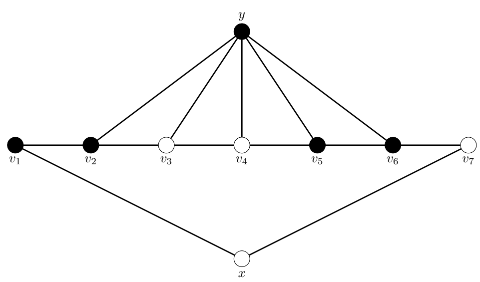

Figure 4: $\gamma_t^e(DS(P_7)) = 5$

Concluding Remarks

The concept of total domination in graphs considers adjacency within the dominating sets while the concept of equitable domination is based on the degree of vertices of dominating sets. The concept of total equitable domination is the combination of both the concepts.

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